§3.3 Poincaré duality

Recall M^n manifold if every point has open nbd homeomorphic to open nbd in $\mathbb{R}^n$.

Note: $H_i(M, M-\{x\}; \mathbb{Z}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathbb{Z})$ (excision).

$\cong \tilde{H}_{i-1}(\mathbb{R}^n - \{0\}; \mathbb{Z})$ ($\mathbb{R}^n$ contractible)

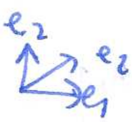
$\cong \tilde{H}_{i-1}(S^{n-1}; \mathbb{Z})$ ($\mathbb{R}^n - \{0\} \simeq S^{n-1}$).

so $H_i(M, M-\{x\}; \mathbb{Z}) = 0$ unless $i = n$, in which case it's $\mathbb{Z}$.

Fact M^n orientable: $H_k(M; \mathbb{Z}) \cong H^{n-k}(M; \mathbb{Z})$

M^n non-orientable: $H_k(M; \mathbb{Z}_2) \cong H^{n-k}(M; \mathbb{Z}_2)$.

Problem: need to define orientation.



$\subseteq \mathbb{R}^3$ ordered basis gives orientation.

Def$^{\underline{n}}$ An orientation of $\mathbb{R}^n$ at x is a choice of generator for $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathbb{Z}) \cong \mathbb{Z}$.

Observation: rotations preserve generator, reflections reverse generator.

Def$^{\underline{n}}$ ~~The~~ A local orientation of M^n at x is a choice of generator for $H_n(M^n, M^n - \{x\};) \mathbb{Z}$

Notation write $H_n(X|A)$ for $H_n(X, X-A)$, so $H_n(X|x)$ means $H_n(X, X-x; \mathbb{Z})$.

want: global orientation to be consistent choice of orientation for all points $x \in M^n$.

Def$^{\underline{n}}$ An orientation of M^n is a function $x \mapsto \mu_x$ assigning to each $x \in M$ a generator $\mu_x \in H_n(M^n|x)$ satisfying local consistency: each $x \in M^n$ is contained in an open ball $\mathbb{R}^n \subset M^n$ s.t. all of the μ_y for $y \in \mathbb{R}^n$ are images of a chosen generator $\mu_B \in H_n(M|B)$ under the map $H_n(M|B) \to H_n(M|y)$.