<u>note</u>: this works when $X=$ point as then $\mu: h^n(X) \to k^n(X)$

is just $\underset{\text{scalar mult.}}{\vee}$ $R \otimes_R H^n(Y;R) \to H^n(Y;R)$

<u>Prop^n</u>: If a natural transformation between unreduced cohomology theories on the category of CW-complexes is an isomorphism for the pair (point, ϕ) then it is an isomorphism for all pairs.

<u>Proof</u> (prop^n, sketch). • 5 lemma, long exact sequence of the pair, suffices to show μ is iso when $A=\phi$.

• X finite dim, do induction on dimension.

– 0-dimensional, holds by hypothesis and axiom for disjoint unions.

– induction step: long exact sequence of pair (X^n, X^{n-1}), 5 lemma $\Rightarrow$ suffices to show that μ is iso for $(X,A) = (X^n, X^{n-1})$.

consider characteristic maps $\Phi: \coprod_\alpha (D^n_\alpha, \partial D^n_\alpha) \to (X^n, X^{n-1})$

excision $\Rightarrow$ Φ^* is an isomorphism for h^*, k^*, naturality implies suffices to show that μ is iso for $(X,A) = \coprod_\alpha (D^n_\alpha, \partial D^n_\alpha)$

disjoint unions $\Rightarrow$ suffices to show for $(D^n_\alpha, \partial D^n)$

5 lemma $\Rightarrow$ result as D^n contractible (0-dimensional case)

∂D^n has dim $(n-1)$ $\square$.

<u>Proof</u> (simple Künneth). need to check h^*, k^*, unreduced cohomology theories and μ is a natural transformation.

1) homotopy invariance: $f \simeq g \Rightarrow f^* = g^*$ ✓.

2) excision: $h^*(X,A) \simeq h^*(B, A\cap B)$ for $X = A\cup B$, CW-subcomplexes ✓.

3) long exact sequence of a pair: k^* ✓.

h^*: take long exact sequence of ordinary cohomology groups for the pair (X,A) and tensor it with the free R-module $H^n(Y;R)$ for fixed n. This gives another long exact sequence as $H^n(Y;R) \cong \bigoplus_\alpha R_\alpha$.

- then let n vary, taking direct sum of then exact sequences, letting n vary to get right dimensions.

4) disjoint unions: k^* ✓

h^*: fact: $\left(\prod_\alpha M_\alpha\right) \otimes_R N \underset{\uparrow}{\cong} \prod_\alpha (M_\alpha \otimes_R N)$ if $\begin{cases} M_\alpha \text{ any } R\text{-modules} \\ N \text{ free } R\text{-module.} \end{cases}$

canonical isomorphism

$N \cong \bigoplus_\beta R_\beta$ so $M_\alpha \otimes_R N = \bigotimes M_{\alpha\beta} = \bigoplus(M_\alpha \otimes_R N_\beta)$ so follows from

$\prod_\beta \prod_\alpha M_{\alpha\beta} \cong \prod_\alpha \prod_\beta M_{\alpha\beta}$ ✓.

<u>μ natural</u> $\Leftarrow$ cup product natural, so natural wrt maps between spaces.

$\Leftarrow$ need to check natural wrt coboundary map, follows from:

$$\begin{array}{ccc} H^k(A;R) \times H^\ell(Y;R) & \xrightarrow{\delta \times 1} & H^{k+1}(X,A;R) \times H^\ell(Y;R) \\ \downarrow \times & & \downarrow \times \\ H^{k+\ell}(A \times Y;R) & \xrightarrow{\delta} & H^{k+\ell+1}(X \times Y, A \times Y; R) \end{array}$$

need to check this at cocycle level. □.

<u>Thm</u> (Relative Künneth) (X,A), (Y,B) CW-pairs.

$$H^*(X,A;R) \otimes_R H^*(Y,B;R) \longrightarrow H^*(X \times Y, A \times Y \cup X \times B; R)$$

ring isomorphism if $H^k(Y;B;R)$ finitely generated free R-module for all k. □.

Examples $\cdot\ \mathbb{C}P^n$: $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\alpha^{n+1}$ $\quad |\alpha|=2$

- $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha]$, $|\alpha|=2$
- complex Grassmannian $G_n(\mathbb{C}^\infty) \leftarrow$ n-dim $\mathbb{C}$-vector subspaces of $\mathbb{C}^\infty$

  $H^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) \cong \mathbb{Z}[\alpha_1,\dots,\alpha_n]$, $|\alpha_i|=2i$.

  $\uparrow$ aka $BU(n)$, classifying space of $U(n)$.

- quaternionic Grassmannian $G_n(\mathbb{H}^\infty) = BSp(n)$

  $H^*(G_n(\mathbb{H}^\infty);\mathbb{Z}) \cong \mathbb{Z}[\alpha_1,\dots,\alpha_n]$ $\quad |\alpha_i|=4i$

- $H^*(BSU(n);\mathbb{Z}) \cong \mathbb{Z}[\alpha_2,\dots,\alpha_n]$ $\quad |\alpha_i|=2i$.

§3.B General Künneth

Thm (general Künneth) X,Y cw-complexes, R principal ideal domain,

$$0 \to \bigoplus_i \left(H_i(X;R)\otimes_R H_{n-i}(Y;R)\right) \to H_n(X\times Y;R) \to \bigoplus_i \mathrm{Tor}\left(H_i(X;R), H_{n-i-1}(Y;R)\right) \to 0$$

$\uparrow$ natural short split exact sequence. $\square$.

ingredients: • homology cross product : $H_i(X;R)\times H_j(Y;R) \to H_{i+j}(X\times Y;R)$

define this in terms of cellular boundary map. note: $I^i\times I^j = I^{i+j}$

Prop^n boundary map in $C_*(X\times Y)$ determined by boundary maps in $C_*(X)$, $C_*(Y)$ by $d(e^i\times e^j) = de^i\times e^j + (-1)^i e^i\times de^j$. $\square$.

Recall : Tor defined in terms of free resolution of modules, can be computed:

1) $\mathrm{Tor}(A,B)\cong \mathrm{Tor}(B,A)$
2) $\mathrm{Tor}(\bigoplus_i A_i, B) = \bigoplus_i \mathrm{Tor}(A_i,B)$
3) $\mathrm{Tor}(A,B)=0$ if A,B torsion free.
4) $\mathrm{Tor}(A,B) = \mathrm{Tor}(T(A),B)$ $\quad T(A)=$ torsion subgroup
5) $\mathrm{Tor}(\mathbb{Z}_n, A) = \ker(A \xrightarrow{\times n} A)$
6) $0\to B\to C\to D\to 0$ exact, then $0\to \mathrm{Tor}(A,B)\to \mathrm{Tor}(A,C)\to \mathrm{Tor}(A,D) \to A\otimes B \to A\otimes C \to A\otimes D \to 0$ exact.