so in particular $H^*(\mathbb{R}P^2;\mathbb{Z}_2) = \{a_0 + a_1 a + a_2 a^2 \quad a_i \in \mathbb{Z}_2\}$. (51)

a_0: dim 0, $a_1 a$: dim 1, $a_2 a^2$: dim 2

with polynomial mult: $(1+a)(1+a^2) = 1 + a^2 + a + a^3 = 1 + a + a^2$.

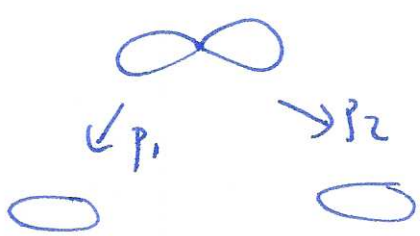
<u>Observation</u> induced homomorphisms are (graded) ring homomorphisms!

from <u>Prop^n</u>: $f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$.

<u>Example/Prop^n</u>. $\widetilde{H}^*(\bigvee_\alpha X_\alpha) \cong \prod_\alpha \widetilde{H}^*(X_\alpha)$.

$\uparrow$ graded ring isomorphism.

<u>Proof/example</u>: $S^1 \vee S^1$



$$\begin{tikzcd} & X_\alpha \vee X_\beta \arrow[dl, "p_\alpha"'] \arrow[dr, "p_\beta"] & \\ X_\alpha & & X_\beta \end{tikzcd}$$

<u>note</u> at the cochain level $p_\alpha^*([z])$ is equal to z on X_α and 0 everywhere else, so each cochain gets extended by zero across $\bigvee_\alpha X_\alpha$,

so $p_\alpha^*([z]) \cup p_\beta^*([y]) = 0$ if $\alpha \neq \beta$. $\square$.

<u>Th^m</u> $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$ if $\alpha \in H^k(X)$, $\beta \in H^l(X)$.

<u>Warning</u> sometimes H^* is just called commutative.

<u>Proof</u> (sketch) $\varphi \cup \psi(\sigma) = \varphi(\sigma|[v_0, \dots v_k])\psi(\sigma|[v_k, \dots v_{k+l}])$.

$\psi \cup \varphi(\sigma) = \psi(\sigma|[v_0, \dots v_l])\varphi(\sigma|[v_l, \dots, v_{k+l}])$.

differ by permutation of the vertices.

observation: if X is a Δ-complex, so is the Δ-complex formed by reversing the order of all the vertices.

given $\sigma : [v_0, \dots, v_n] \to X$ let $\bar{\sigma} : [v_n, \dots v_0] \to X$.

this requires $n+(n-1)+\dots+1 = \frac{1}{2}n(n+1)$ swaps of vertices.

define $\epsilon_n = (-1)^{\frac{1}{2}n(n+1)}$

define $\rho: C_n(X) \to C_n(X)$

$\sigma \mapsto \epsilon_n \bar{\sigma}$

<u>claim</u>: ρ is a chain map, chain homotopic to the identity.

claim $\Rightarrow$ Thm:
$$(\rho^*\phi \cup \rho^*\psi)(\sigma) = \phi(\epsilon_k \sigma|[v_k, \dots v_0])\,\psi(\epsilon_\ell \sigma|[v_{k+\ell} \dots v_k])$$

$$\rho^*(\psi\cup\phi) = \epsilon_{k+\ell}\,\psi(\sigma|[v_{k+\ell} \dots v_k)\,\phi(\sigma|[v_k, \dots v_0]).$$

so
$$\epsilon_k\epsilon_\ell(\rho^*\phi\cup\rho^*\psi) = \epsilon_{k+\ell}\,\rho^*(\psi\cup\phi)$$

<u>check</u> $\epsilon_k\epsilon_\ell = (-1)^{k\ell}\epsilon_{k+\ell}$. $\quad (-1)^{\frac{1}{2}k(k+1)+\frac{1}{2}\ell(\ell+1)} \quad \epsilon^{\frac{1}{2}(k+\ell)(k+\ell+1)} =$

$$= \epsilon^{\frac{1}{2}k(k+1)+\frac{1}{2}\ell(\ell+1)+\frac{1}{2}k\ell+\frac{1}{2}\ell k} = \epsilon_k\epsilon_\ell(-1)^{\ell k}.$$

ρ^* identity on homology $\Rightarrow$ $[\phi]\cup[\psi] = (-1)^{k\ell}[\psi]\cup[\phi]$. $\square$.

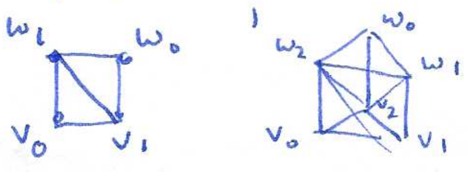
<u>check</u> • ρ is a chain map: $\partial\rho = \rho\partial$

$$\partial\rho(\sigma) = \epsilon_n \sum (-1)^i \sigma|[v_n, \dots \hat{v}_{n-i} \dots v_0]$$
$$\rho\partial(\sigma) = \rho\left(\sum(-1)^i\sigma|[v_0, \dots \hat{v}_i, \dots v_n]\right)$$
$$= \underbrace{\epsilon_{n-1}(-1)^{n-i}}\sigma|[v_n, \dots, \hat{v}_{n-i} \dots v_0])$$

check $\epsilon_n = \epsilon_{n-1}(-1)^n$

• construct chain homotopy $P: C_n(X) \to C_{n+1}(X)$ $\qquad \pi: \Delta^n\times I \to \Delta^n$

$$P(\sigma) = \sum_i (-1)^i \epsilon_{n-i}(\sigma\pi)|[v_0, \dots v_i, w_n, \dots w_i].$$



claim: this works $\square$.

<u>Defⁿ</u> Exterior algebra $\Lambda_R[\alpha_1, \alpha_2 \dots] \leftarrow \overset{R}{\text{commutative ring w/ identity}}$

is the free R-module with basis finite products $\alpha_{i_1} \dots \alpha_{i_k}$

$i_1 < i_2 < \dots < i_k$ with $\alpha_i \alpha_j = -\alpha_j \alpha_i$ and $\alpha_i^2 = 0$

empty product $\leftrightarrow 1$.

<u>Fact</u> $H^*(T^n; R) = \Lambda_R[\alpha_1, \dots, \alpha_n]$.

<u>Q:</u> is $H^*(X \times Y) = H^*(X) \times H^*(Y)$ A: it's complicated.

<u>Recall</u> cross product / external cup product

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

$$H^*(X) \times H^*(Y) \xrightarrow{\times} H^*(X \times Y)$$

$$(a, b) \longmapsto a \times b = p_1^*(a) \cup p_2^*(b).$$

cup products distributive $\Rightarrow \times$ is a bilinear map $\leftarrow$ might not even be a homomorphism! Solution: replace $H^*(X;R) \times H^*(Y;R)$ with $H^*(X;R) \otimes H^*(Y;R)$.

$\uparrow$ e.g. $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$
$(a,b) \mapsto ab$.

<u>Tensor products</u> <u>A, B abelian groups</u>

$A \otimes B$ abelian group, with generators: $a \otimes b$ for $a \in A$, $b \in B$

relations: $(a + a') \otimes b = a \otimes b + a' \otimes b$

$a \otimes (b + b') = a \otimes b + a \otimes b'$

so $0 \in A \otimes B$ is $0 = 0 \otimes 0 = 0 \otimes a = a \otimes 0$ $\leftarrow$ check:

$\underbrace{(a+0)}_{a} \otimes b = a \otimes b + 0 \otimes b$
$\Rightarrow 0 \otimes b = 0$

and $-(a \otimes b) = (-a) \otimes b = a \otimes (-b)$

useful properties: 1) $A \otimes B \cong B \otimes A$

2) $(\bigoplus A_i) \otimes B = \bigoplus (A_i \otimes B)$

3) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

4) $\mathbb{Z} \otimes A = A$

5) $\mathbb{Z}_n \otimes A = A/nA$

6) $f: A \to A'$, $g: B \to B'$ homomorphisms induce $f \otimes g: A \otimes B \to A' \otimes B'$ homomorphism

by $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$

7) $\phi: A \times B \to C$ bilinear map gives a homomorphism

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] & C \\ A \otimes B \arrow[ur, "\Phi"'] & \end{tikzcd}$$

such that $a \otimes b \mapsto \phi(a,b)$.

can define $A \otimes B$ in term of this universal property

<u>observation</u> can now calculated $A \otimes B$ for finitely generated abelian grps

e.g. $\mathbb{Z}^a \otimes \mathbb{Z}^b \cong \mathbb{Z}^{ab}$

<u>A, B R-modules</u> define $A \otimes_R B$ generators: $a \otimes b$ for $a \in A, b \in B$

relations: $(a+a') \otimes b = a \otimes b + a' \otimes b$

$a \otimes (b+b') = a \otimes b + a \otimes b'$

extra: $ra \otimes b = a \otimes rb$ for all $r \in R, a \in A, b \in B$.

$\Rightarrow$ $A \otimes_R B$ is now an R-module.

<u>warning</u> $A \otimes_R B$ depends on R! $\left[\; R = \mathbb{Q}(\sqrt{2}) \quad \underbrace{R \otimes_R R}_{\dim_{\mathbb{Q}} 2} \neq \underbrace{R \otimes_{\mathbb{Q}} R}_{\dim_{\mathbb{Q}} 4} \right]$.

<u>useful facts</u> as above, but 4): $R \otimes_R A \cong R$.

<u>consequence</u>:

$$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X \times Y; R)$$

$$a \otimes b \longmapsto a \times b.$$

define multiplication in graded rings by $\boxed{(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd}$

<u>check</u>: $(a\times b)(c\times d) = p_1^*(a)\cup p_2^*(b)\cup p_1^*(c)\cup p_2^*(d)$

$= (-1)^{|b||c|}\, p_1^*(a)\cup p_1^*(c)\cup p_2^*(b)\cup p_2^*(d)$

$= (-1)^{|b||c|}\, p_1^*(a\cup c)\cup p_2^*(b\cup d)$.

$= (-1)^{|b||c|}\, (ac)\times(bd)$. $\square$.

<u>Thm</u> (simple Künneth) $H^*(X;R)\otimes_R H^*(Y;R)\to H^*(X\times Y;R)$
is a ring isomorphism if
- X, Y CW-complexes
- $H^k(Y;R)$ finitely generated free R-module $\forall k$.

<u>Examples</u> ① $H^*(\mathbb{R}P^\infty\times \mathbb{R}P^\infty;\mathbb{Z}_2)\cong H^*(\mathbb{R}P^\infty;\mathbb{Z}_2)\otimes H^*(\mathbb{R}P^\infty;\mathbb{Z}_2)$

$\cong \mathbb{Z}_2[\alpha]\otimes\mathbb{Z}_2(\beta)\cong\mathbb{Z}_2[\alpha,\beta]$.

<u>Defn</u>
exterior algebra $\Lambda_R[\alpha_1,\alpha_2,\dots]$ ← graded tensor product over R of $\Lambda_R[\alpha_i]$, where $|\alpha_i|=1$.

② $H^*(\underbrace{S^1\times S^1\times\dots\times S^1}_{T^n};\mathbb{Z})\cong\Lambda_{\mathbb{Z}}[\alpha_1,\dots,\alpha_n]$.

<u>Proof</u> (Künneth, sketch) fix CW complex Y, and consider functors

$h^n(X,A) = \bigoplus_i \left(H^i(X,A;R)\otimes_R H^{n-i}(Y;R)\right)$

$k^n(X,A) = H^n(X\times Y, A\times Y;R)$

cross product defines a map $\mu: h^n(X,A)\to k^n(X,A)$

want: μ is an isomorphism when $X=$ CW complex, $A=\emptyset$.