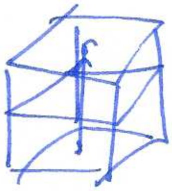
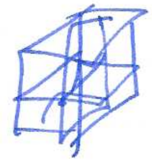
cup product is dual to the intersection form $H^a(M^n)\times H^b(M^n)\to H^{\overset{a+b}{\cancel{n-a-b}}}_{\cancel{n-a-b}}(M^n)$.

Example T_3:  $H_1 \times H_2 \xrightarrow{\ \cap\ } H_0$  $H_2\times H_2 \xrightarrow{\ \cap\ } H_1$

Warning: cup product / intersection form cannot depend only on the (co)homology groups: Fact there are manifolds with same H_k, H^k, but different cup/cap products. So we need a definition at the chain level.

Defn cup product $C^k(X;R)\times C^l(X;R)\to C^{k+l}(X;R)$

$$\phi\ ,\ \psi \longmapsto \phi\cup\psi$$

by $(\phi\cup\psi)(\sigma)=\phi(\sigma|_{[v_0,\dots,v_k]})\,\psi(\sigma|_{[v_k,\dots,v_{k+l}]})$.

↑ product in R.

Q: does this induce a map on cohomology $H^k\times H^l\to H^{k+l}$?

Lemma $\delta(\phi\cup\psi)=\delta\phi\cup\psi+(-1)^k\phi\cup\delta\psi$

Proof $\sigma:\Delta^{k+l+1}\to X$

$$(\delta\phi\cup\psi)(\sigma)=\sum_{i=0}^{k+1}(-1)^i\phi(\sigma|[v_0,\dots,\hat v_i,\dots v_{k+1}])\,\psi(\sigma|[v_{k+1},\dots,v_{k+l+1}])$$

$$(-1)^k(\phi\cup\delta\psi)(\sigma)=\sum_{i=k}^{k+l+1}(-1)^i\phi(\sigma|[v_0,\dots,v_k])\,\psi(\sigma|[v_k,\dots,\hat v_i,\dots,v_{k+l+1}])$$

↑ add these two terms - last term of first sum cancels first term of second sum, remaining terms exactly $\delta(\phi\cup\psi)=\phi\cup\psi(\partial\sigma)$ □.

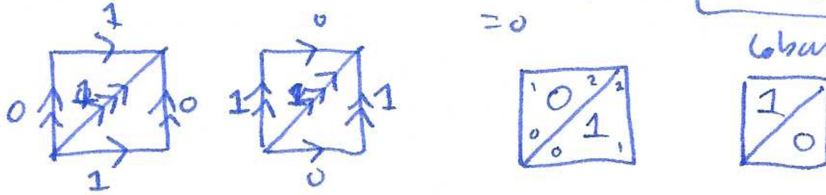
Check: $H^k(X)\times H^l(X)\to H^{k+l}(X)$.

$$[\phi]\quad [\psi]\ \mapsto\ [\phi\cup\psi].$$

- product of two cocycles is a cocycle: $\delta(\phi\cup\psi) = \underbrace{\delta\phi}_{=0}\cup\psi + (-1)^k\phi\cup\underbrace{\delta\psi}_{=0} = 0.$ ✓.

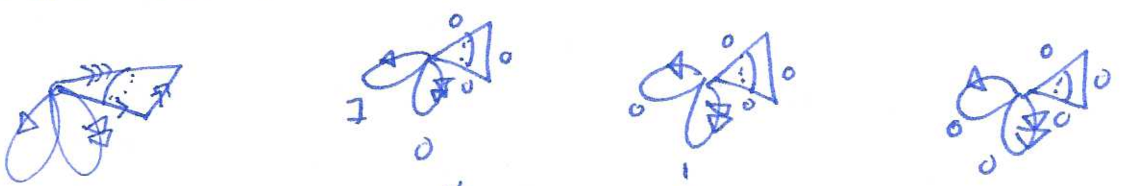
- product of cocycle and coboundary is a coboundary:
  - if $\phi = \delta\alpha$ then $\phi\cup\psi = \delta\alpha\cup\psi = \pm\,\alpha\cup\underbrace{\delta\psi}_{0} - \underbrace{\delta(\alpha\cup\psi)}_{\text{coboundary}}$
  - if $\psi = \delta\beta$ then $\phi\cup\psi = \phi\cup\delta\beta = \pm\,\underbrace{\delta\phi}_{=0}\cup\beta - \underbrace{\delta(\phi\cup\beta)}_{\text{coboundary}}$ ✓.

<u>Examples</u> • $T^2 = S^1\times S^1$



α β $\alpha\cup\beta$ $\beta\cup\alpha$

• $X = S^1\vee S^1\vee S^2$



α β $\alpha\cup\beta = 0 = \beta\cup\alpha$.

<u>Fact</u> chain level defn gives relative cup products

$$\left.\begin{array}{l} H^k(X)\times H^l(X,A) \xrightarrow{\cup} H^{k+l}(X,A) \\ H^k(X,A)\times H^l(X) \xrightarrow{\cup} H^{k+l}(X,A) \\ H^k(X,A)\times H^l(X,A) \xrightarrow{\cup} H^{k+l}(X,A) \end{array}\right\}$$

if ϕ (or ψ) $= 0$ on A so does any product.

more generally

$$H^k(X,A)\times H^l(X,B) \xrightarrow{\cup} H^{k+l}(X, A\cup B)$$ ← use subdivision and fact above.

<u>Propn</u> $f: X\to Y$ cts induces $f^*: H^k(Y)\to H^k(X)$ s.t.

$f^*(\alpha\cup\beta) = f^*(\alpha)\cup f^*(\beta)$, also in relative cases.

i.e. f^* preserves cup products.

<u>Proof</u>: at the chain level: $f^\#(\phi)\cup f^\#(\psi) = f^\#(\phi\cup\psi)$, as:

$$(f^\# \phi \cup f^\# \psi)(\sigma) = f^\# \phi(\sigma|[v_0,\dots,v_k])\, f^\# \psi(\sigma|[v_k,\dots,v_{k+\ell}])$$

$$= \phi(f\sigma|[v_0,\dots,v_k])\, \psi(f\sigma|[v_k,\dots,v_{k+\ell}])$$

$$= (\phi \cup \psi)(f\sigma) = f^\# (\phi\cup\psi)(\sigma). \quad \square.$$

Def$^{\underline{n}}$ cross product (sometimes called external cup product)

$$H^k(X) \times H^\ell(Y) \longrightarrow H^{k+\ell}(X\times Y)$$

$$H^k(X,A) \times H^\ell(Y,B) \to H^{k+\ell}(X\times Y,\ A\times Y \cup X\times B)$$

$$(\alpha\ ,\ \beta) \longmapsto \alpha\times\beta = p_1^*(\alpha) \cup p_2^*(\beta)$$

where

$$\begin{tikzcd} & X\times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}.$$

The cohomology ring ← graded ring, i.e. remember dimensions.

Def$^{\underline{n}}$ $H^*(X) = \bigoplus H^n(X)$ ← elements are finite sums $\Sigma \alpha_i$ with $\alpha_i \in H^i(X)$.

with $(\Sigma_i \alpha_i)(\sum_j \beta_j) = \sum_{i,j} \alpha_i \beta_j$

identity is $1 \in H^0(X)$, map all 0-simplices to 1.

In some cases, this ring is easy to describe.

Th$^{\underline{m}}$ $H^*(\mathbb{R}P^n;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/\alpha^{n+1}$

$H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$. $\quad |\alpha| = 1$

$H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^{n+1}$

$H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[\alpha] \quad |\alpha| = 2. \quad \square$