§3.2 Cup products.

Intuition (for manifolds, cup products and duality).

Defn M^n is an n-manifold if every point $x \in M^n$ has an open neighborhood homeomorphic to an open ball in $\mathbb{R}^n$ (+ Hausdorff, 2nd countable)

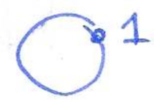
Recall . homology class $\leftrightarrow$ closed k-dim subset of M^n, which doesn't bound a $(k+1)$-dim subset. $\partial z = 0$, $z \neq \partial y$
$[z] \in H_k(M)$

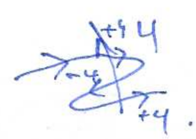
Examples  

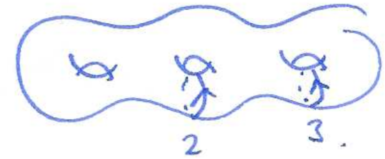
• cohomology class $\leftrightarrow$ function on k-dim subsets of M^n, cocycle, i.e. $\delta f = 0$.
$[f] \in H^k(M)$ i.e. $f(z) = 0$ if $z = \partial y$
but f not a coboundary, i.e. $f \neq \delta g$ for $g \in H^{k-1}(M^n)$
i.e. f does not arise from function on $(k-1)$-dim subspaces.

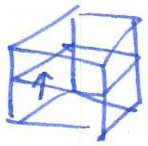
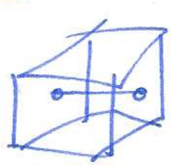
Fact (Poincare duality) Every $[f] \in H^k(M^n)$ is dual to some $[z] \in H^{n-k}(M)$

intuition: cohomology comes from counting intersections.

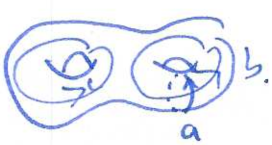
Examples • S^1 $H^0(S^1) \cong \mathbb{Z}$ dual to $H_1(S^1)$ 
$H^1(S^1) \cong \mathbb{Z}$ dual to $H_0(S^1)$ 

• $S^1 \times S^1$: $H^0(S^1) \cong \mathbb{Z}$
$H^1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z} \leftarrow$  
$H^2(S^1) \cong \mathbb{Z}$.

• S_2 : $H^1(S^2) \cong \mathbb{Z}^4$  • S_3 

• $T^3 = S^1 \times S^1 \times S^1$  $\leftrightarrow H^1$. H_2  $H_1 \leftrightarrow H^2$.

Fact there is an intersection form on homology: $H_a(M^n) \times H_b(M^n) \to H_{a+b-n}(M^n)$

Examples S_2: 
$H_1 \times H_1 \to H_0$
$a \cap b \mapsto 1$
$a \cap c \mapsto 0$