~~Facts~~ if C is free then every exact sequence $0 \to A \to B \to C \to 0$ splits.

pick basis $\{c_\alpha\}$ for C and then define $s(c_\alpha) = b_\alpha$ for some choice of $b_\alpha \in B$ s.t. $j(b_\alpha) = c_\alpha$.

Observation suppose $r : X \to A$ is a retraction, so $ri = \mathbb{1}$. $\underbrace{A \overset{i}{\hookrightarrow} X \xrightarrow{r} A}_{\mathbb{1}}$

so $\underbrace{H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{r_*} H_n(A)}_{\mathbb{1}}$ $\Rightarrow$ i_* injective, r_* surjective

consider long exact seq for (X,A). $\cdots H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X,A) \to H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X)$ ← injective

so breaks up into $0 \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \to 0$, with $r_* : H_n(X) \to H_n(A)$

and $r_* i_* = \mathbb{1}_* : H_n(A) \to H_n(A)$

so short exact sequence splits, i.e. $H_n(X) \cong H_n(A) \oplus H_n(X,A)$.

can be used to rule out some retractions...

Homology of groups (taste).

G group $\rightsquigarrow$ CW complex $K(G,1)$ ← Eilenberg–MacLane space. s.t. $\pi_1(K(G,1)) = G$

universal cover $\widetilde{K(G,1)}$ contractible.

Fact homotopy type of $K(G,1)$ depends only on G.

Observation $H_n(K(G,1))$ depends only on G, and so are group invariants.

sometimes write $H_n(G)$.

Warning if G has torsion then $K(G,1)$ has ∞ dim, so $H_n(G) \neq 0$ possible for all n.

Homology w/ coefficients

we've consider chains $\sum n_i \sigma_i$ $n_i \in \mathbb{Z}$. $C_n(X)$

can consider chains $\sum n_i \sigma_i$ $n_i \in G$ ← abelian gp. $C_n(X;G)$

then form chain complex: $\cdots \to C_n(X;G) \xrightarrow{\partial_n} C_n(X;G) \to \cdots$

same formula $\partial \sum n_i \sigma_i = \sum (-1)^n n_i \sigma_i | [v_0, \ldots \hat{v}_i \ldots v_n]$.

resulting homology groups are $H_n(X;G)$, $H_n(X,A;G)$ ← homology gps w/ coeffs in G.

Special case $\mathbb{Z}_2$: don't need to worry about signs, computationally fast, useful for non-orientable manifolds.

Fact: this works for cellular homology too.

Lemma: If $f: S^k \to S^k$ has deg n, then $f_*: H_k(S^k; G) \to H_k(S^k; G)$ is multiplication by n. $\square$.

Example $\mathbb{RP}^n$, $\mathbb{Z}_2$ coeffs. recall. $\cdots \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{2} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0$

all maps 0, so $H_k(\mathbb{RP}^n) = \mathbb{Z}_2 \quad 0 \le k \le n$
0 else.

§2.3 Formal viewpoint

Axioms for homology $\quad$ special case: $X =$ CW complex, reduced homology.

Def$^{\underline{n}}$ A reduced homology theory assigns to each CW complex X a sequence of abelian groups $\tilde{h}_n(X)$, and to each map $f: X \to Y$ a sequence of homomorphisms $f_*: \tilde{h}_n(X) \to \tilde{h}_n(Y)$. such that $\mathbb{1}_* = \mathbb{1}$ and $(fg)_* = f_* g_*$ and:

1) if $f \simeq g: X \to Y$ then $f_* = g_*: \tilde{h}_n(X) \to \tilde{h}_n(Y)$

2) there are boundary homomorphisms $\partial: \tilde{h}_n(X/A) \to \tilde{h}_{n-1}(A)$ defined for each CW pair (X, A) fitting in to a long exact seq.

$$\cdots \xrightarrow{\partial} \tilde{h}_n(A) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(X/A) \xrightarrow{\partial} \tilde{h}_{n-1}(A) \to \cdots$$

$\uparrow$ inclusion $\qquad \uparrow$ quotient

and these boundary maps are natural, i.e. if $f: (X, A) \to (Y, B)$ then

$$\begin{array}{ccc} \tilde{h}_n(X/A) & \xrightarrow{\partial} & \tilde{h}_{n-1}(A) \\ \downarrow \bar{f}_* & & \downarrow f_* \\ \tilde{h}_n(Y/B) & \xrightarrow{\partial} & \tilde{h}_{n-1}(B) \end{array}$$

commutes.

3) for a wedge sum $X = \bigvee_\alpha X_\alpha$ with inclusions $i_\alpha: X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ the direct sum map $\bigoplus_\alpha i_\alpha: \bigoplus_\alpha \tilde{h}_n(\alpha) \to \tilde{h}_n(\bigvee_\alpha X_\alpha)$. is an isomorphism $\forall n$

Facts. • ok if non-zero for $-$ve n. • ok if $\tilde{h}_n(\{pt\}) \neq 0$!

• can give similar axioms for unreduced homology, but slightly more awkward.

• the coefficients of the homology theory are $\tilde{h}_n(x_0) = \tilde{h}_n(S^0)$.

• if h is a homology theory defined for CW-pairs and $h_n(x_0) = 0$ for all $n \neq 0$, then $h_n(X,A) \cong H_n(X;A;G)$, when $G = h_0(x_0)$.

• a homology theory is a functor from (CW-complexes, cts maps) to (graded abelian gps, homomorphisms).

§2A Homology and fundamental group

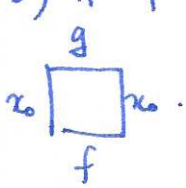
Thm $H_1(X) = ab(\pi_1 X)$.

More precisely, we can regard loops based at x_0 as singular 1-cycles, and this gives a map $h: \pi_1(X, x_0) \to H_1(X)$. If X is path connected, this is surjective, and the kernel is the commutator subgroup.

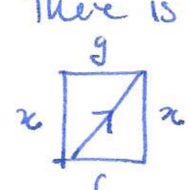
Proof notation: $f \simeq g$ homotopy fixing endpoints.
$f \sim g$ homologous chains, i.e. $f - g$ is a boundary.

1) If f is a constant path, then $f \sim 0$. In fact f is the boundary of the constant 2-simplex $\sigma : [v_0, v_1, v_2] \to x_0$ as $\partial \sigma = \sigma|[v_1,v_2] - \sigma|[v_0,v_2] + \sigma|[v_0,v_1] = f - f + f = f$.

2) If $f \simeq g$ then $f \sim g$. $f \simeq g$ means there is a map $F: I \times I \to X$



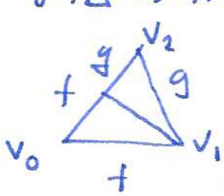
subdivide square into 2-simplices:



✓.

3) $f \cdot g \sim f + g$
path composition; sum in chain group

$\sigma : \Delta^2 \to X$, project orthogonally onto $[v_0, v_2]$ and then do $f \cdot g$



$\partial \sigma = f + g - f \cdot g$

4) $\bar{f} \sim -f$, when $\bar{f}$ is inverse path for f, so $f + \bar{f} \sim f \cdot \bar{f} \sim 0$.

2),3) $\Rightarrow$ there is a well defined homomorphism h: loops $\longrightarrow$ 1-cycles
$h: \pi_1(X, x_0) \longrightarrow H_1(X)$.

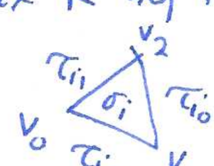
__claim__ X path connected, then h surjective.

__Proof__ let $\sum n_i \sigma_i$ be a 1-cycle in $C_1(X)$ representing $[\alpha] \in H_1(X)$. up to relabelling, assume $n_i = \pm 1$. use 4) wlog each $n_i = +1$, so just have $\sum \sigma_i$. Note: $\partial(\sum \sigma_i) = 0$, so if some σ_i is not a loop, there must be some σ_j s.t. $\sigma_i(v_1) = \sigma_j(v_0)$ so $\sigma_i \cdot \sigma_j$ is well defined. 3) $\Rightarrow$ can replace ~~$\sigma_i \sigma_j$~~ $\sigma_i + \sigma_j$ by $\sigma_i \cdot \sigma_j$. repeat until every σ_i is a loop. X path connected $\Rightarrow \exists$ path $\gamma_i : x_0 \to \sigma_i(v_0)$ for all i, and 3),4) $\Rightarrow$ $\gamma_i \cdot \sigma_i \cdot \bar{\gamma}_i \sim \sigma_i$ so can assume all loops are based at x_0, so $[\alpha]$ lies in image of h.

~~claim~~ __note__ commutator subgroup $\subseteq \ker(h)$ as $H_1(X)$ abelian.

__claim__ $\ker(h) \leq$ commutator subgroup of $\pi_1(X, x_0)$.

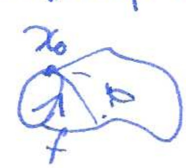
__Proof__ suffices to show every $[f] \in \ker(h)$ is trivial in $\pi_1(X)_{ab}$.
suppose $[f] \in \ker(h)$, then f is the boundary of a 2-chain $\sum n_i \sigma_i$
as before can assume $n_i = \pm 1$, can build 2-dim Δ-complex K by taking a Δ^2 for each σ_i and identifying pairs of faces



$$\partial \sigma_i = \tau_{i0} - \tau_{i1} + \tau_{i2}$$

$$f = \partial\left(\textstyle\sum_i n_i \sigma_i\right) = \sum_i n_i \partial \sigma_i = \sum_{i,j} (-1)^j n_i \tau_{ij}$$

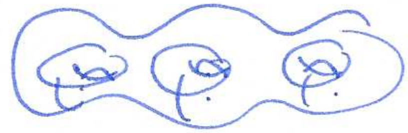
can group all but one of the edges into pairs for which the coefficients are $+1$ and -1, remaining $\tau_{ij} = f$. Gives map $\sigma: K \to X$
can homotopy σ s.t. all vertices have image x_0 as X path connected, so now all edges loops at x_0.



now in $\pi_1(X, x_0)_{ab}$: $[f] = \sum_{ij} (-1)^j n_i [\tau_{ij}] = \sum_i n_i [\partial \sigma_i]$

where $[\partial \sigma_i] = [\tau_{i0}] - [\tau_{i1}] + [\tau_{i2}]$, but σ_i gives null homotopy of the compound loop $\tau_{i0} - \tau_{i1} + \tau_{i2} \Rightarrow [f] = 0$ in $\pi_1(X)_{ab}$. $\square$.

Fact in proof can choose K to be an orientable surface w/ 1 boundary component.

Example. $H_1(S_g)$  $\leftarrow$ generators.

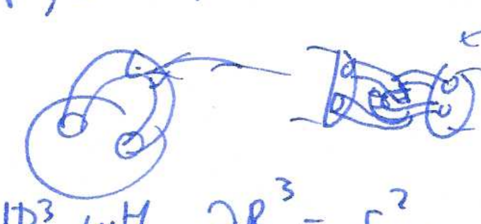
§2.B Classical applications

Prop^n^ a) If $D \subset S^n$ and D homeomorphic to D^k, $0 \leq k \leq n$, then $\tilde{H}_i(S - D)$ is zero for all i.

b) If $S \subseteq S^n$ is homeomorphic to S^k for some $0 \leq k \leq n$ then $\tilde{H}_i(S^n - S)$ is $\mathbb{Z}$ for $i = n-k-1$, and 0 otherwise.

Proof: induction / MV $\square$.

Warning recall Jordan curve theorem $S^1 \subset \mathbb{R}^2$ separates $\mathbb{R}^2$ into two components 
b) says $S^{n-1} \subset S^n$ separates it into two components, with same homology as $\{pt\}$.
but complements may not be simply connected.

Example Alexander Horned sphere.  $\leftarrow$ replace inside each tube.

claim this gives map $f: B^3 \subset \mathbb{R}^3$ with $\partial B^3 = S^2$.

claim $\pi_1(\mathbb{R}^3 \setminus B)$ has trivial abelianization, but is non-trivial.

Division algebras

Thm $\mathbb{R}, \mathbb{C}$ are the only (finite dim) division algebras over $\mathbb{R}$ which are commutative and have an identity.