

Defn the cellular homology groups $H_n^{CW}(X)$ are the homology groups of this chain complex.

Thm If X is a CW-complex, then $H_n^{CW}(X) = H_n(X)$.

Proof ~~injective~~, so $H_n(X) = H_n(X^n) / \text{im } \partial_{n+1}$

$H_n(X^{n+1}) \cong H_n(X)$ and map surjective, so $H_n(X) \cong H_n(X^n) / \text{im } \partial_{n+1}$

j_n injective $\Rightarrow \text{im}(\partial_{n+1})$ gets mapped to $\text{im}(j_n \partial_{n+1}) = \text{im}(\partial_{n+1})$

so $H_n(X^n)$ gets mapped isomorphically to $\text{im}(j_n) = \text{ker}(\partial_n)$

j_{n-1} injective $\Rightarrow \text{ker } \partial_n = \text{ker}(d_n)$. So j_n gives isomorphism of $H_n(X^n) / \text{im } \partial_{n+1} \cong \frac{\text{ker}(d_n)}{\text{im}(d_{n+1})}$. $\square$

Applications 1) $H_n(X) = 0$ if X has no n -cells.

2) if X has at most k n -cells then $H_n(X)$ generated by at most k elements.

4) If X has no cells in adjacent dimensions, then $H_n(X)$ is free abelian with generators corresponding to cells.

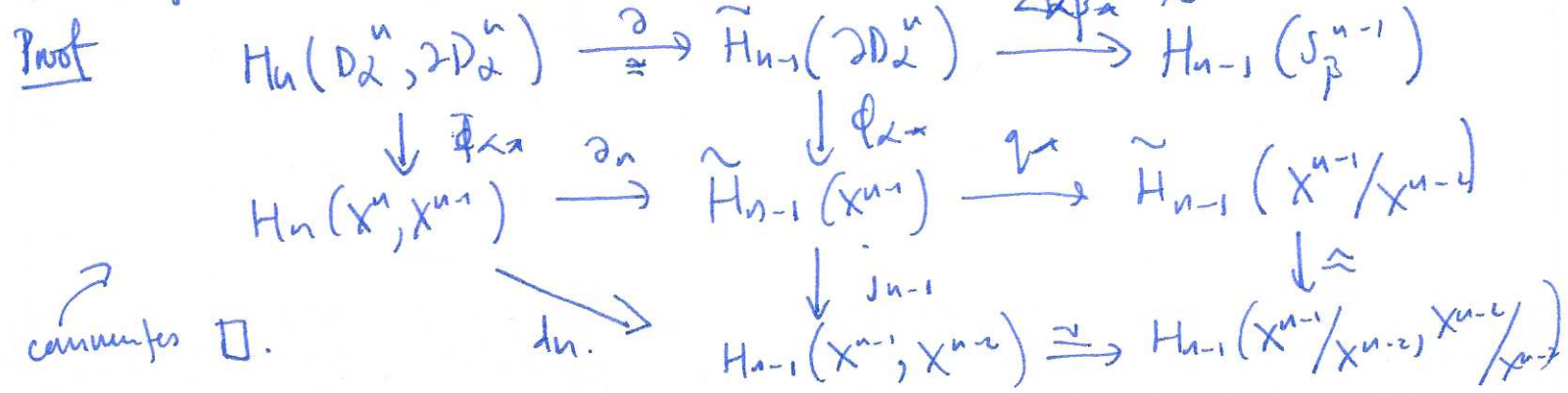
Example $\mathbb{CP}^n \leftarrow$ fact has one cell in each even dim $\leq 2n$.

so $H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k=0,2,\dots,2n \\ 0 & \text{otherwise} \end{cases}$

how to compute boundary maps

Cellular boundary formula

$d_n(e_\alpha^n) = \sum_{\beta} d_{\alpha\beta} e_\beta^{n-1}$, where $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{\psi_\beta} S_\beta^{n-1}$ that is the composition of the attaching map of e_α^n with the quotient map collapsing $X^{n-1} - e_\beta^{n-1}$ to a point.



Example ① genus g surface S_g

1	0-cell
$2g$	1-cells
1	2-cell

attaching map $[a_1, b_1] \dots [a_g, b_g]$

$$0 \rightarrow \mathbb{Z} \xrightarrow{\quad} \mathbb{Z} \xrightarrow{\quad} \mathbb{Z} \rightarrow 0 \Rightarrow H_k(S_g) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^{2g} & k=1 \\ \mathbb{Z} & k=2 \\ 0 & k \geq 3 \end{cases}$$

② Moore spaces $\tilde{H}_k(X) = \begin{cases} \mathbb{Z} & \text{for } k=n \\ 0 & \text{else} \end{cases}$

let $X = S^n \cup_{\phi} \mathbb{R}P^{n+1}$ where $\phi: \partial \mathbb{R}P^{n+1} \rightarrow S^n$ has degree d .

then chain complex is

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times d} \mathbb{Z} \rightarrow 0 \dots \rightarrow \mathbb{Z} \rightarrow 0 \quad \square$$

③ real projective space $\mathbb{R}P^n \leftarrow 1$ cell in each dimension, attaching maps $\phi: S^{k-1} \rightarrow \mathbb{R}P^{k-1}$.

two sheeted covering maps

so chain is

$$0 \rightarrow \mathbb{Z} \xrightarrow{\quad} \mathbb{Z} \xrightarrow{\quad} \mathbb{Z} \xrightarrow{\quad} \dots \xrightarrow{\quad} \mathbb{Z} \xrightarrow{\quad} \mathbb{Z} \xrightarrow{\quad} \mathbb{Z} \rightarrow 0$$

compute degrees of maps:

$$S^{k-1} \xrightarrow{\phi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1} / \mathbb{R}P^{k-2} = S^{k-2}$$

$S^{k-1} \rightarrow S^{k-2}$ has two components which are S^{k-1}/S , ϕ is a homeomorphism restricted to each component, so an $x \in S^{k-1} \setminus S^{k-2}$ has two pre-images, one in each ball, and these two maps differ from each other by composition with the antipodal map on S^{k-1} , which has degree $(-1)^{k-1}$, so

$$\deg(q\phi) = \deg(1) + \deg(\overset{\text{antipodal map}}{-1}) = 1 + (-1)^k = \begin{cases} 0 & k \text{ odd} \\ 2 & k \text{ even} \end{cases}$$

$$\text{so } H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k=0, k=n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd}, 0 < k < n \\ 0 & \text{else} \end{cases}$$

Euler characteristic

X finite CW complex: $\chi(X) = \sum_n (-1)^n c_n$ $c_n = \# \text{cells in dimension } n.$

Thm $\chi(X) = \sum_n (-1)^n \text{rank}(H_n(X)).$

useful fact: $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ short exact then $\text{rank}(B) = \text{rank}(A) + \text{rank}(C).$

Proof (algebra). let $0 \rightarrow C_n \xrightarrow{d_n} C_{n-1} \rightarrow \dots \rightarrow C_1 \xrightarrow{d_1} C_0 \rightarrow 0$ be a chain complex of finitely generated ^{abelian} groups, with cycles $Z_n = \ker(d_n)$, boundaries $B_n = \text{Im}(d_{n+1})$, homology $H_n = Z_n / B_n$

this gives short exact sequences $0 \rightarrow Z_n \rightarrow C_n \rightarrow B_{n-1} \rightarrow 0$

and $0 \rightarrow B_n \rightarrow Z_n \rightarrow H_n \rightarrow 0$

so $\begin{cases} \text{rank } C_n = \text{rank } Z_n + \text{rank } B_{n-1} \\ \text{rank } Z_n = \text{rank } B_n + \text{rank } H_n \end{cases} \Rightarrow \text{rank } C_n = \text{rank } B_n + \text{rank } H_n + \text{rank } B_{n-1}$
 add up these with alternating signs in each dim.

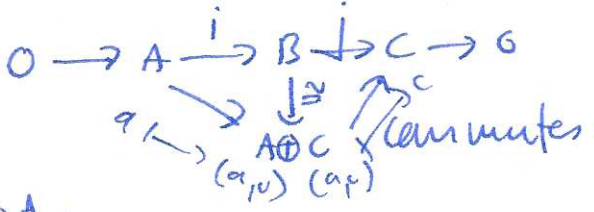
get $\sum (-1)^n \text{rank } C_n = \sum (-1)^n \text{rank } H_n \quad \square.$

Example surfaces $S_g \quad \chi(S_g) = 1 - 2g + 1 = 2 - 2g$

Split exact sequences

Splitting Lemma $0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$ short exact TFAE:

- 1) there is an isomorphism $B \cong A \oplus C$ s.t.
- 2) there is a homomorphism $p: B \rightarrow A$ s.t. $p \circ i = \text{id}_A$.
- 3) there is a homomorphism $s: C \rightarrow B$ s.t. $j \circ s = \text{id}_C$.



Proof (sketch) $2) \Rightarrow 1)$ check $B \rightarrow A \oplus C$ $b \mapsto (p(b), j(b))$ is an isomorphism w/ desired properties.
 $3) \Rightarrow 1)$ check $A \oplus B \rightarrow B$ $(a, b) \mapsto i(a) + s(b)$ is an isomorphism w/ desired properties.
 $1) \Rightarrow 2), 1) \Rightarrow 3)$ define p, s as obvious maps $\square.$