Defn The cellular homology groups $H_n^{CW}(X)$ are the homology groups of this chain complex.

Thm: If X is a CW-complex, then $H_n^{CW}(X) = H_n(X)$.

Proof ~~j_n injective, so $H_n(X) = H_n(X^n)/\operatorname{im}\partial_{n+1}$~~

$H_n(X^{n+1}) \cong H_n(X)$ and map surjective, so $H_n(X) \cong H_n(X^n)/\operatorname{im}\partial_{n+1}$

j_n injective $\Rightarrow$ $\operatorname{im}(\partial_{n+1})$ gets mapped to $\operatorname{Im}(j_n\partial_{n+1}) = \operatorname{im}(d_{n+1})$

so $H_n(X^n)$ gets mapped isomorphically to $\operatorname{Im}(j_n) = \ker(\partial_n)$

j_{n-1} injective $\Rightarrow$ $\ker\partial_n = \ker(d_n)$. So j_n gives isomorphism of $H_n(X^n)/\operatorname{im}\partial_{n+1} \cong \ker(d_n)/\operatorname{Im}(d_{n+1})$

$\square$.

Applications 1) $H_n(X) = 0$ if X has no n-cells.

2) if X has at most k n-cells then $H_n(X)$ generated by at most k elements.

4) If X has no cells in adjacent dimensions, then $H_n(X)$ is free abelian with generators corresponding to cells.

Example $\mathbb{C}P^n$ ← fact has one cell in each even dim $\leq 2n$.

so $H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & k = 0, 2, \dots 2n \\ 0 & \text{else} \end{cases}$

how to compute boundary maps

Cellular boundary formula $d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$, where $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \to X^{n-1} \to S_\beta^{n-1}$ that is the composition of the attaching map of e_α^n with the quotient map collapsing $X^{n-1} - e_\beta^{n-1}$ to a point.

Proof

$$\begin{tikzcd}
H_n(D_\alpha^n, \partial D_\alpha^n) \arrow[r, "\partial", "\cong"'] \arrow[d, "\Phi_{\alpha *}"] & \tilde H_{n-1}(\partial D_\alpha^n) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \tilde H_{n-1}(S_\beta^{n-1}) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \tilde H_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \tilde H_{n-1}(X^{n-1}/X^{n-2}) \arrow[d, "\cong"] \\
& H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\cong"] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{tikzcd}$$

commutes $\square$.

<u>Example</u> ① genus g surface S_g 1 0-cell
2g 1-cells
1 2-cell attaching map $[a_1,b_1]\cdots[a_g,b_g]$

$$0 \to \mathbb{Z} \xrightarrow[0]{} \mathbb{Z}^{2g} \xrightarrow[0]{} \mathbb{Z} \to 0 \Rightarrow H_k(S_g) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^{2g} & k=1 \\ \mathbb{Z} & k=2 \\ 0 & k \geq 3 \end{cases}$$

② <u>Moore spaces</u> $\tilde{H}_k(X) = \begin{cases} \mathbb{Z}/d\mathbb{Z} & \text{for } k=n \\ 0 & \text{else} \end{cases}$

let $X = S^n \cup_\phi e^{n+1}$ where $\phi: \partial e^{n+1} \to S^n$ has degree d.

then chain complex is

$$\overset{n+1}{} \qquad \overset{n}{}$$
$$0 \to \mathbb{Z} \xrightarrow[\times d]{} \mathbb{Z} \to 0 \cdots \to \mathbb{Z} \to 0. \quad \square$$

③ <u>real projective space $\mathbb{RP}^n$</u> ← 1 cell in each dimension, attaching maps two sheeted covering maps $\phi: S^{k-1} \to \mathbb{RP}^{k-1}$.

so chain is

$$\overset{n}{} \quad \overset{n-1}{} \qquad\qquad \overset{2}{} \quad \overset{1}{} \quad \overset{0}{}$$
$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

compute degrees of maps: $S^{k-1} \xrightarrow{\phi} \mathbb{RP}^{k-1} \xrightarrow{q} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} = S^{k-1}$.

$S^{k-1} - S^{k-2}$ has two components which are B^{k-1}'s, ϕ is a homeomorphism restricted to each component, so an $x \in S^{k-1} \setminus S^{k-2}$ has two pre-images, one in each ball, and these two maps differ from each other by composition with the antipodal map on S^{k-1}, which has degree $(-1)^k$, so

$$\deg(q\phi) = \deg(\mathbb{1}) + \deg(\underset{\substack{\uparrow \\ \text{antipodal map}}}{-\mathbb{1}}) = 1 + (-1)^k = \begin{cases} 0 & k \text{ odd.} \\ 2 & k \text{ even} \end{cases}$$

so $H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k=0,\ k=n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd},\ 0<k<n \\ 0 & \text{else.} \end{cases}$

Euler characteristic

X finite CW complex : $\chi(X) = \sum_n (-1) c_n$ $\qquad c_n =$ # cells in dimension n.

Thm $\chi(X) = \sum_n (-1) \operatorname{rank}(H_n(X))$.

useful fact: $0 \to A \to B \to C \to 0$ short exact then $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$.

Proof (algebra). Let $0 \to C_k \xrightarrow{d_k} C_{k-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$ be a chain complex of finitely generated ~~groups~~ abelian groups, with cycles $Z_n = \ker(d_n)$.

boundary $B_n = \operatorname{Im}(d_{n+1})$

homology $H_n = Z_n / B_n$

this gives short exact sequences $0 \to Z_n \to C_n \to B_{n-1} \to 0$

and $0 \to B_n \to Z_n \to H_n \to 0$

so $\left.\begin{array}{l} \operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1} \\ \operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n \end{array}\right\}$ $\operatorname{rank} C_n = \operatorname{rank} B_n + \operatorname{rank} H_n + \operatorname{rank} B_{n-1}$

add up these with alternating signs in each dim.

get $\sum (-1)^n \operatorname{rank} C_n = \sum (-1)^n \operatorname{rank} H_n$ $\square$.

Example surfaces S_g $\chi(S_g) = 1 - 2g + 1 = 2 - 2g$

Split exact sequences

$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ short exact TFAE:

Splitting lemma

1) there is an isomorphism $B \cong A \oplus C$ s.t.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[dr, "a \mapsto (a{,}0)"'] & B \arrow[r, "j"] \arrow[d, "\cong"] & C \arrow[r] & 0 \\ & & A \oplus C \arrow[ur, "(a{,}c) \mapsto c"'] & & \end{tikzcd}$$

commutes

2) there is a homomorphism $p: B \to A$ s.t. $pi = \mathbb{1}: A \to A$.

3) there is a homomorphism $s: C \to B$ s.t. $js = \mathbb{1}: C \to C$.

Proof (sketch) 2) $\Rightarrow$ 1) check $\begin{array}{c} B \to A \oplus C \\ b \mapsto (p(b), j(b)) \end{array}$ is an isomorphism w/ desired properties.

3) $\Rightarrow$ 1) check $\begin{array}{c} A \oplus B \to B \\ (a,c) \mapsto i(a) + s(c) \end{array}$ is an isomorphism w/ desired properties.

1) $\Rightarrow$ 2), 1) $\Rightarrow$ 3) define p, s as obvious maps $\square$.