<u>Useful properties</u> • $\deg(\mathbb{1}_{S^n}) = 1$ as $\mathbb{1}_{\mathbb{Z}} = 1 \mapsto 1$.

• $\deg(f) = 0$ if f not surjective, so map factor $S^n \to S^n - \{x\} \to S^n$ (contracts).
$H_n(S^n)$ $H_n(\text{contr})$ H_n
$\mathbb{Z}$ 0 $\mathbb{Z}$.

• $f \simeq g \Rightarrow \deg(f) = \deg(g)$.

<u>Thm</u> (Hopf) If $f, g : S^n \to S^n$ and $\deg(f) = \deg(g) \Rightarrow f \simeq g$. □.

• $\deg(fg) = \deg(f)\deg(g)$ as $(fg)_* = f_* g_*$. $\Rightarrow$ if f h.e. then $\deg(f) = \pm 1$.

• f reflection on S^n in S^{n-1} $\Rightarrow \deg(f) = -1$.

• antipodal map $\alpha : S^n \to S^n \subseteq \mathbb{R}^{n+1}$, $x \mapsto -x$ has degree $(-1)^{n+1}$ as composition of $(n+1)$ reflections.

• if $f : S^n \to S^n$ has no fixed points, then f homotopic to antipodal map

but $f_t(x) = \dfrac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$.

<u>Thm</u> S^n has a continuous field of non-zero tangent vectors iff n is odd ?

<u>Proof</u> 

can normalize so $|v(x)| = 1$, wt $x \perp v(x)$.
consider homotopy $f_t(x) = \cos(t)x + (\sin t)v(x)$, gives homotopy from $\mathbb{I}$ to antipodal map $-\mathbb{I}$ $\Rightarrow \deg(\mathbb{1}_{S^n}) = 1 = \deg(-\mathbb{I}) = (-1)^{n+1} \Rightarrow n$ odd.

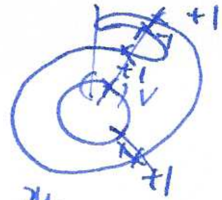
<u>Converse</u>: $n = 2k-1$ set $v(x_1, \dots x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$.
then $x \cdot v(x) = 0$, so non-zero tangent field on S^n.
wt $S^{2k-1} \subset \mathbb{C}^k$, multiplication by i. □.

<u>Prpn</u> $\mathbb{Z}_2$ is the only non-trivial gp that can act freely on S^n if n even.

<u>Proof</u> $\deg(\text{homeo}) = \pm 1$, so $G \curvearrowright S^n$ gives map $d : G \to \{\pm 1\}$ homomorphism.
If action is free, then every non-trivial element of $G \mapsto (-1)^{n+1}$
if n even $\Rightarrow \ker(d) = 1 \Rightarrow G = \mathbb{Z}_2$ □.

Local computation of degree

example



Let $f: S^n \to S^n$ and $y \in S^n$

assume $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_k$.

Let V be small nbd of y, and let U_i be small nbds of x_i s.t. $f(U_i) \subset V$.

by excision: $f: H_n(S^n, S^n - x_i) \longrightarrow H_n(S^n, S^n - y)$

$$\begin{array}{ccc} H_n(S^n, S^n - x_i) & \xrightarrow{f} & H_n(S^n, S^n - y) \\ \cong & & \cong \\ H_n(U_i, U_i - x_i) & & H_n(V, V - y) \\ \cong & & \cong \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ 1 & \xrightarrow{\deg(f|x_i)} & \deg(f|x_i)\mathbb{Z} \end{array}$$

Prop^n $\deg(f) = \sum_i \deg(f|x_i)$.

Proof

$$\begin{array}{ccccc} & & H_n(U_i, U_i - x_i) & \xrightarrow{f_*} & H_n(V, V-y) \\ & \swarrow \cong & \downarrow k_i \text{ inclusion} & & \downarrow \cong \\ H_n(S^n, S^n - x_i) & \xleftarrow{p_i} & H_n(S^n, S^n - f^{-1}(y)) & \xrightarrow{f_*} & H_n(S^n, S^n - y) \\ & \nwarrow \cong & \uparrow j & & \uparrow \cong \\ & & H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \end{array}$$

fact everything commutes, all groups on outside $\mathbb{Z}$.

- p_i projection ~~so p_i~~ onto i-th factor, commute $\Rightarrow p_i j(1) = 1$.

so $j(1) = (1, 1, \ldots, 1) = \sum k_i(1)$ ~~$\xrightarrow{f_*} \deg(f)$~~

top $f_* = \deg f|x_i \Rightarrow$ middle f_* sends $(1, 1, \ldots, 1)$ to $\sum_i \deg(f|x_i)$

$\Rightarrow$ lower f_* is $\deg(f_*)$, so $\deg(f_*) = \sum_i \deg(f|x_i)$ $\square$.

useful fact

Prop^n $\deg(Sf) = \deg(f)$, where $Sf: S^{n+1} \to S^{n+1}$ is suspension of $f: S^n \to S^n$.

Proof CS^n  S^n. So $CS^n / S^n = SS^n = S^{n+1}$

$f: S^n \to S^n$ gives $Cf: (CS^n, S^n) \to (CS^n, S^n)$

with quotient $Sf: S^{n+1} \to S^{n+1}$

$$\begin{array}{ccc} \tilde{H}_{n+1}(S^{n+1}) & \xrightarrow{\partial} & \tilde{H}_n(S^n) \\ \downarrow Sf_* & & \downarrow f_* \\ \tilde{H}_{n+1}(S^{n+1}) & \xrightarrow{\partial} & H_n(S^n) \end{array}$$

commutes from naturality of boundary map in long exact sequence of pair (CS^n, S^n) $\square$.

§ Cellular homology . useful facts about cell complexes :

Lemma X cw-complex, then

a) $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$ and is free abelian for $k=n$. w/ basis $\Leftrightarrow$ n-cells of X^n.

b) $H_k(X^n) = 0$ for $k > n$, in particular if $\dim(X) = n$ then $H_k(X) = 0$ for $k > d$.

c) the inclusion $i : X^n \hookrightarrow X$ gives an isomorphism $i_* : H_k(X^n) \to H_k(X)$ if $k < n$.

Proof a) : (X^n, X^{n-1}) good pair $X^n / X^{n-1} \cong \bigvee_\alpha S^n_\alpha$ one S^n for each cell.

b) consider l.e.s of pair (X^n, X^{n-1}) :

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1})$$

if $k \neq n, n-1$, then two outer groups are equal, so $H_k(X^{n-1}) = H_k(X^n)$ for $k \neq n, n-1$

so for $k > n$ $H_k(X^n) = H_k(X^{n-1}) = \dots H_k(X^0) = 0$.

c) if $k < n$ then $H_k(X^n) = H_k(X^{n+1}) = \dots = H_k(X^{n+m})$ for all $m \geq 0$.

∞-dim case needs a bunch of work $\square$.

Cellular chain complex : $\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$

consider long exact seq of pairs (X^{n+1}, X^n), (X^n, X^{n-1}), (X^{n-1}, X^{n-2}) :

$$H_n(X^{n-1}) = 0 \to H_n(X^n) \to H_n(X^{n+1}) \cong H_n(X) \to \cdots$$

$$\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}} H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$$

$$\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$$

$$H_{n-1}(X^{n-2}) = 0 \to H_{n-1}(X^{n-1}) \to \cdots$$

Def^n $d_n = j_{n-1} \partial_n$

claim $d_n d_{n+1} = 0$.