Example $T^2 =$  $=$  $\quad A \cap B$  $\simeq S^1 \sqcup S^1$.

$X \qquad A \quad B$

$\simeq S^1 \quad \simeq S^1$

$$\cdots \to H_2(A\cap B) \to H_2(A) \oplus H_2(B) \to H_2(X) \to$$
$$0 \qquad\qquad 0 \qquad 0 \qquad \mathbb{Z}$$

$$\to H_1(A\cap B) \to H_1(A) \oplus H_1(B) \to H_1(X) \to$$
$$\mathbb{Z}\oplus\mathbb{Z} \qquad \mathbb{Z} \qquad \mathbb{Z} \qquad \mathbb{Z}\oplus\mathbb{Z}.$$
$$(a,b) \longmapsto (a+b, -a+b).$$

$$\to H_0(A\cap B) \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0.$$
$$\mathbb{Z}\oplus\mathbb{Z} \qquad \mathbb{Z} \qquad \mathbb{Z}. \qquad \mathbb{Z}$$
$$(a,b) \longmapsto (a+b, -(a+b)).$$

ker $\mathbb{Z}$.

Equivalence of simplicial and singular homology

Thm $A \subset X$ Δ-complex pair, then $H_n^\Delta(X,A) \xrightarrow{\cong} H_n(X,A)$ are isomorphisms for all n.

Proof (special case X finite dim $A = \emptyset$).

let X^k be the k-skeleton of X, consider pair (X^k, X^{k-1}).

$$\begin{array}{ccccccccc}
H^\Delta_{n+1}(X^k, X^{k-1}) & \to & H^\Delta_n(X^{k-1}) & \to & H_n(X^k) & \to & H^\Delta_n(X^k, X^{k-1}) & \to & H^\Delta_{n-1}(X^{k-1}) \\
\downarrow ① & & \downarrow ② & & \downarrow ③ & & \downarrow ④ & & \downarrow ⑤ \\
H_{n+1}(X^k, X^{k-1}) & \to & H_n(X^{k-1}) & \to & H_n(X^k) & \to & H_n(X^k, X^{k-1}) & \to & H_{n-1}(X^{k-1})
\end{array}$$

(commutes).

①, ④ isomorphisms for all n: because $(X^k, X^{k-1}) \simeq \bigvee_\alpha S^k$ and we know explicit generators for $H_k(S^k)$. Induction on $k \Rightarrow$ ② ⑤ isomorphism. result follows from 5 lemma $\square$.

Thm 5 Lemma If the two rows are exact, and $\alpha, \beta, \delta, \epsilon$ isomorphisms, then γ is an isomorphism:

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\
\downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \downarrow\delta & & \downarrow\epsilon \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E'
\end{array}$$

<u>Proof</u> (diagram chase)

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E' \end{tikzcd}$$

$c \mapsto d \mapsto l(d)$

$b' \mapsto c' - \gamma(c)$, $\quad c' \mapsto k'(c') \mapsto l'k'(c) = 0$.

<u>γ surjective</u>

Start with $c' \in C'$,

$k'(c') \in D'$, ~~δ surjective~~

δ surjective $\Rightarrow \exists\, d \in D$ s.t. $\delta(d) = k'(c')$

~~diagram~~

ϵ injective, diagram commutes $\Rightarrow \epsilon(l(d)) = 0 \Rightarrow l(d) = 0$.

top row exact $\Rightarrow \exists\, c \in C$ s.t. $k(c) = d$.

consider $c' - \gamma(c) \in C'$, $k'(c' - \gamma(c)) = k'(c') - k'\gamma(c) = k'(c') - \delta k(c) = k'(c') - k'(c') = 0$

so $\exists\, b' \in B'$ s.t. $j'(b') = c' - \gamma(c)$

β surjective $\Rightarrow \exists\, b \in B$ s.t. $\beta(b) = b'$.

<u>claim</u>: $\gamma(c + j(b)) = \gamma(c) + \gamma j(b) = \gamma(c) + j'\beta(b) =$ ~~$k(d) + j'(b')$~~ $\gamma(c) + j'(b') = \gamma(c) + c' - \gamma(c) = c'$ as required.

<u>γ injective</u>

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E' \end{tikzcd}$$

$b \mapsto c \mapsto k(c)$, $\quad a' \mapsto \beta(b)$, $\quad 0 \mapsto k'(0) = 0$

assume $\gamma(c) = 0$

δ injective $\Rightarrow \delta k(c) = 0 \Rightarrow k(c) = 0$

so $\exists\, b \in B$ s.t. $j(b) = c$

$j'\beta(b) = 0$, so $\beta(b) = i'(a')$ for some $a' \in A'$

α surjective $\Rightarrow \exists\, a \in A$ s.t. $\alpha(a) = a'$.

β injective $\Rightarrow \beta(i(a) - b) = \beta i(a) - \beta(b) = i'\alpha(a) - \beta(b) = i'(a') - \beta(b) = 0$

$\Rightarrow i(a) - b = 0 \Rightarrow b = i(a)$, so $c = j(b) = ji(a) = 0$ as required $\square$.

<u>Observation / defn</u> If X has a Δ-complex (CW) structure w/ finitely many n-cells, then $H_n(X)$ is finitely generated, so $H_n(X) \cong \mathbb{Z}^b \oplus \mathbb{Z}_{a_1} \oplus \dots \oplus \mathbb{Z}_{a_k}$.

b = n-th <u>Betti number</u>. a_i <u>torsion coefficients</u>.

§2.2 Applications

<u>§ Degree</u>. $f: S^n \to S^n$ induces $f_*: H_n(S^n) \to H_n(S^n)$

$$\mathbb{Z} \to \mathbb{Z}$$
$$1 \mapsto d(f) \leftarrow \text{\underline{degree} of } f.$$

<u>Defn</u>.