Fact long exact seq of pair/triple is natural i.e. if $f:(X,A)\to(Y,B)$, then

$$\begin{array}{ccccccccc}\cdots \to & H_n(A) & \to & H_n(X) & \to & H_n(X,A) & \xrightarrow{\partial} & H_{n-1}(A) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow f_* & \\ \cdots \to & H_n(B) & \to & H_n(Y) & \to & H_n(Y,B) & \to & H_{n-1}(B) & \to \cdots \end{array}$$

commutes.

Mayer-Vietoris sequence Thm $A,B\subset X$, $A\cup B = X$. then

$$\cdots \to H_n(A\cap B) \xrightarrow{\Phi} H_n(A)\oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_n(A\cap B) \to \cdots$$

is a long exact seq.

Proof let $C_n(A+B) \subset C_n(X)$ be chains which are sum of chains in A and chains in B.

note ∂ map sends $C_n(A+B)\to C_{n-1}(A+B)$, so forms a chain complex.

subdivision result $\Rightarrow$ $C_n(A+B) \hookrightarrow H_n(X)$ induces an isomorphism on homology.

Claim: $0\to C_n(A\cap B) \xrightarrow{\phi} C_n(A)\oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0$

$x \longmapsto (x,-x)$

$(x,y) \longmapsto x+y$

is short exact given

check. $C_n(A\cap B) \hookrightarrow C_n(A)$ so injective. ✓.

$C_n(A)\oplus C_n(B) \to C_n(A+B)$ surjective ✓.

$\psi\phi = 0$ ✓.

$\ker(\psi) = \mathrm{im}(\phi)$: spose $\psi(x,y) = 0$, then $x+y=0$ in $C_n(A+B)$

so $x=-y$ in $C_n(A+B)$ $\Rightarrow$ x,y both in $C_n(A\cap B)$, s. $\exists\, x\in C_n(A\cap B)$

st $\phi(x) = (x,-x) = (x,y)$. $\square$.

now: short exact sequence of chain complexes gives long exact seq in homology:

$$\cdots \to H_n(A\cap B) \xrightarrow{\Phi} H_n(A)\oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A\cap B) \to \cdots$$

$[x] \longmapsto [x], [-x]$.

$([x],[y]) \longmapsto [x]-[y]$.

Q: what does ∂ do?

let $[z]\in H_n(X)$, can write $z = x+y$ where $x\in$ ~~H_n~~ $C_n(A)$, $y\in C_n(B)$

$\partial z = 0 \Rightarrow \partial x = -\partial y$, ~~so send~~ so $\partial x \in C_n(A\cap B)$, send $z \mapsto \partial x$ $\square$.