recall: Thm (Excision) $Z \subset A \subset X$, $\overline{Z} \subset \mathring{A}$, then $(X-Z, A-Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X-Z, A-Z) \xrightarrow{\cong} H_n(X, A)$ for all n. Equivalently, if $A, B \subset X$ with $\mathring{A} \cup \mathring{B} = X$, then $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphism $H_n(B, A\cap B) \to H_n(X, A)$ for all n.

Proof (2nd version) $X = A \cup B$, set $\mathcal{U} = \{A, B\}$.

notation: $C_n(A+B)$ for $C_n^{\mathcal{U}}(X)$. recall: there are maps D, ρ s.t. $\partial D + D \partial = \mathbb{1} - i\rho$ and $\rho i = \mathbb{1}$ (*) ← all of these maps take chains in A to chains in A, so induce maps on quotients by quotienting out $C_n(A)$, and then quotient maps also satisfy (*)

~~This gives~~ So this map $C_n(A+B)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$ induces an isomorphism on homology. Furthermore: $C_n(B)/C_n(A \cap B) \longrightarrow C_n(A+B)/C_n(A)$

is an isomorphism, as each chain group is isomorphic, as they have the same basis, i.e simplices lying in $B \setminus A$.

Therefore $H_n(B, A\cap B) \cong H_n(X, A)$, as required $\square$.

version for quotient spaces X/A:

Propⁿ (X, A) good pair. The quotient map $q: (X, A) \to (X/A, A/A)$ induces isomorphisms $q_*: H_n(X, A) \to H_n(X/A, A/A) \cong \tilde{H}_n(X)$ for all n.

Proof $A \subset V \subset X$, where V is an open neighbourhood that deformation retracts onto A. claim: follow diagram commutes:

$$\begin{tikzcd}
H_n(X, A) \arrow[r] \arrow[d, "q_*"] & H_n(X, V) \arrow[d, "q_*"] & H_n(X \setminus A, V \setminus A) \arrow[l] \arrow[d, "q_*"] \\
H_n(X/A, A/A) \arrow[r] & H_n(X/A, V/A) & H_n(X/A \setminus A/A, V/A - A/A) \arrow[l]
\end{tikzcd}$$

$\cong$ from long exact sequence of the triple (X, V, A) as $H_n(V,A)=0$ for all n as $(V,A)\simeq(A,A)$, and $H_n(A,A)=0$ for all n.

excision $\Rightarrow \cong$

$$\begin{array}{ccccc} H_n(X,A) & \longrightarrow & H_n(X,V) & \xleftarrow{\cong} & H_n(X-A, V-A) \\ \downarrow q_* & & \downarrow q_* & & \downarrow q_* \\ H_n(X/A, A/A) & \longrightarrow & H_n(X/A, V/A) & \longleftarrow & H_n(X/A - A/A, V/A - A/A) \end{array}$$

iso as $q|_{X-A}$ is a homeo.

isomorphism from long exact sequence of triples for (X, V, A) as $(V,A)\simeq(A,A)$ $\Rightarrow (V/A, A/A)\simeq(A/A, A/A)$

excision $\Rightarrow \cong$.

then commutativity $\Rightarrow$ the vertical maps are isos. $\square$.

<u>Applications</u>

① explicit generators for $H_n(D^n, \partial D^n)$ and $H_n(S^n)$.

- $H_n(D^n, \partial D^n)$ <u>claim</u>: $i_n : \Delta^n \to \Delta^n$ generates $H_n(D^n,\partial D^n)\cong\mathbb{Z}$.

induction: $n=0$ ✓

induction step: let $\Lambda \subset \Delta^n$ be a union of all but one $(n-1)$-dim face of Δ^n. There are isomorphisms

$$H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\cong} H_{n-1}(\partial\Delta^n, \Lambda) \leftarrow H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$$

boundary map in l.e.s of triple $(\Delta^n, \partial\Delta^n, \Lambda)$ as $H_i(\Delta^n, \Lambda)=0$ for all i

preceding Prop^n as good pair and $\Delta^{n-1} \overset{i}{\hookrightarrow} \partial\Delta^n$ as face not in Λ gives a homeo of quotients $\Delta^{n-1}/\partial\Delta^{n-1} \cong \partial\Delta^n/\Lambda$

this sends

$$i_n \longmapsto \partial i_n = \pm i_{n-1} \text{ in } C_{n-1}(\partial\Delta^n, \Lambda)$$

- $H_n(S^n)$ let $S^n = \Delta_1^n \cup \Delta_2^n$, identify boundaries by order preserving linear homeo. Then $\Delta_1^n - \Delta_2^n$ is a cycle, claim: generates

$H_n(S^n)$ (assuming $n>0$) consider

$$\tilde{H}_n(S^n) \xrightarrow{\cong} H_n(S^n, \Delta_2^n) \xleftarrow{\cong} H_n(\Delta_1^n, \partial\Delta_1^n)$$

↑ long exact seq of pair (S^n, Δ_2^n) ↑ pass to quotients as before

$$\Delta_1^n - \Delta_2^n \longleftrightarrow \Delta_1^n$$ ← which is a generator by previous example.

<u>Corollary</u> If the CW complex X is the union of two CW-subcomplexes A, B then $(B, A\cap B) \overset{i}{\hookrightarrow} (X, A)$ induces an isomorphism
& $A\cap B \neq \emptyset$ $\quad H_n(B, A\cap B) \cong H_n(X, A)$.

<u>Proof</u> CW pairs are good, so can pass to quotient $B/_{A\cap B} \cong X/A$ $\square$.

<u>Corollary</u> For a wedge sum $\bigvee_\alpha X_\alpha$ the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism $\bigoplus_\alpha i_{\alpha*} : \bigoplus \tilde{H}_n(X_\alpha) \to \tilde{H}_n(\bigvee_\alpha X_\alpha)$ provided all the pairs (X_α, x_α) are good.

<u>Proof</u> take $(X, A) = (\bigsqcup_\alpha X_\alpha, \bigsqcup \{x_\alpha\})$ $\square$.

<u>Thm</u> [Brouwer] (Invariance of domain) If two non-empty open sets $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ are homeomorphic then $m=n$.

<u>Proof</u> let $x \in U$, then $H_k(U, U-\{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus x)$ by excision. Long exact sequence of the pair $(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$ gives

$$H_k(\mathbb{R}^m, \mathbb{R}^m \setminus x) \cong \tilde{H}_{k-1}(\mathbb{R}^m - \{x\})$$

↑ deformation retracts onto S^{m-1}

so $\tilde{H}_k(U, U\setminus x) \cong \begin{cases} \mathbb{Z} & k=m \\ 0 & k\neq m \end{cases}$, same argument for $U, V \Rightarrow m=n$. $\square$.