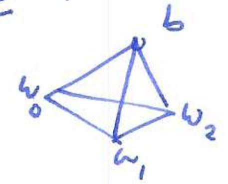
special case: $Y \subseteq \mathbb{R}^m$ convex, consider linear maps $\Delta^n \to Y$.

gives linear chains: $LC_n(Y) < C_n(Y)$. note $\partial LC_n(Y) \subset LC_{n-1}(Y)$, so $LC_n(Y)$ forms a chain complex, augment, set $LC_{-1}(Y) := \mathbb{Z} = \langle[\phi]\rangle$ with $\partial[v_0] = [\phi]$. Let λ be the linear map $\lambda: \Delta^n \to [w_0, \dots w_n]$ where $w_i = \lambda(v_i)$.

Each point $b \in Y$ determines a homomorphism $b: LC_n(Y) \to LC_{n+1}(Y)$ given by $b([w_0, \dots, w_n]) = [b, w_0, \dots, w_n]$, i.e. cone off each linear simplex on b. <u>Note</u>: $\partial b([w_0, \dots, w_n]) = [w_0, \dots, w_n] - b(\partial[w_i])$ for each simplex, extend linearly: $\partial b(\alpha) = \alpha - b(\partial\alpha)$ for all $\alpha \in LC_n(Y)$.

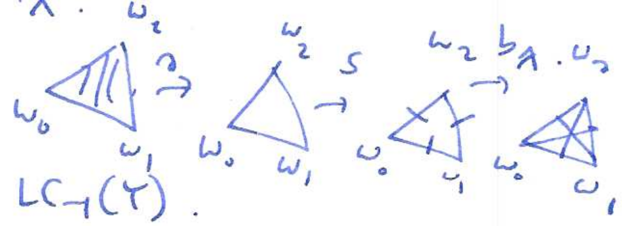
equivalent to $\partial b + b\partial = \mathbb{1}$.



so b is a chain homotopy between between $\mathbb{1}$ and 0 on $LC(Y)$.

Define a <u>subdivision homomorphism</u> $S: LC_n(Y) \to LC_n(Y)$ inductively as follows: let $\lambda: \Delta^n \to Y$ be a basis element, let b_λ be the image of the barycenter of Δ^n under λ.

Then set $S(\lambda) = b_\lambda(S\partial\lambda)$ ← i.e.



check this works: define $S[\phi] = [\phi]$, so id on $LC_{-1}(Y)$.

- $n=0$: $S([w_0]) = w_0(S\partial[w_0]) = w_0(S[\phi]) = w_0([\phi]) = [w_0]$.
- $n=1$: etc...

<u>Claim</u> $\partial S = S\partial$, i.e. S is a chain map on $LC(Y)$.

<u>note</u> $n=0,1$: $S = \mathbb{1}$ so ok ✓.

$n \geq 2$:

$$
\begin{aligned}
\partial S\lambda &= \partial(b_\lambda(S\partial\lambda)) && \text{use: } \partial b_\lambda + b_\lambda\partial = \mathbb{1}. \\
&= S\partial\lambda - b_\lambda(\partial S\partial\lambda) \\
&= S\partial\lambda - b_\lambda(S\partial\partial\lambda) && \leftarrow \text{induction on } n \\
&= S\partial\lambda && \leftarrow \text{as } \partial^2 = 0.
\end{aligned}
$$

now build a chain homotopy $T: LC_n(Y) \to LC_{n+1}(Y)$ from S to $\mathbb{1}$:

$$\begin{tikzcd}
\cdots \arrow[r] & LC_{n+1}(Y) \arrow[r] \arrow[d, "S"', "\mathbb{1}"] & LC_n(Y) \arrow[r] \arrow[d, "S"', "\mathbb{1}"] \arrow[dl, "T"'] & LC_{n-1}(Y) \arrow[d, "S"', "\mathbb{1}"] \arrow[dl, "T"'] \\
\cdots \arrow[r] & LC_{n+1}(Y) \arrow[r] & LC_n(Y) \arrow[r] & LC_{n-1}(Y)
\end{tikzcd}$$

define T by: $n=-1$: set $T=0$ on $LC_{-1}(Y) = \mathbb{Z}$.

$n \geq 0$: $T\lambda = b_\lambda(\lambda - T\partial\lambda)$

$\Delta^n \times I$ $\leftarrow$ add barycenter to $\Delta^n \times \{1\}$.
then cone off simplices in $\Delta^n \times \{0\} \cup \partial\Delta^n \times I$
then project $\Delta^n \times I \to \Delta^n$.

<u>claim</u> $\partial T + T\partial = \mathbb{1} - S$

<u>$n=-1$</u> : $T=0$, $S=\mathbb{1}$.

<u>$n \geq 0$</u> : $\partial T\lambda = \partial(b_\lambda(\lambda - T\partial\lambda))$ use $\partial b_\lambda = \mathbb{1} - b_\lambda \partial$

$= \lambda - T\partial\lambda - b_\lambda(\partial(\lambda - T\partial\lambda))$

$= \lambda - T\partial\lambda - b_\lambda(S\partial\lambda + T\partial\partial\lambda)$ induction on n.

$= \lambda - T\partial\lambda - S\lambda$ as $\partial^2 = 0$, $S\lambda = b_\lambda(S\partial\lambda)$.

<u>2) barycentric subdivision of general chains</u>

<u>Defn</u> $S: C_n(X) \to C_n(X)$ by $S\sigma = \sigma_\# S\Delta^n$

claim: this is a chain map: $\partial S\sigma = \partial\sigma_\# S\Delta^n$

$= \sigma_\# \partial S\Delta^n$

$= \sigma_\# S\partial\Delta^n$ i-th face of Δ^n

$= \sigma_\# S\left(\sum_i (-1)^i \Delta_i^n\right)$

$= \sum_i (-1)^i \sigma_\# S\Delta_i^n$

$= \sum_i (-1)^i S(\sigma|\Delta_i^n)$

$= S\left(\sum_i (-1)^i \sigma|\Delta_i^n\right) = S(\partial\sigma)$.

Similarly, define $T: C_n(X) \to C_{n+1}(X)$ by $T\sigma = \sigma_\# T\Delta^n$,

claim : $\partial T + T\partial = 1 - S$, i.e. gives chain homotopy between 1 and S.

check: $\partial T\sigma = \partial \sigma_\# T\Delta^n = \sigma_\# \partial T\Delta^n = \sigma_\# (\Delta^n - S\Delta^n - T\partial\Delta^n)$

$$= \sigma - S\sigma - \sigma_\# T\partial\Delta^n$$

$$= \sigma - S\sigma - T(\partial\sigma).$$

4) Iterated barycentric subdivision let S^m be m-th barycentric subdivision

claim : $1, S^m$ are chain homotopic by $D_m = \sum_{i=0}^{m-1} TS^i$

check : $\partial D_m + D_m\partial = \sum_{i=0}^{m-1} (\partial TS^i + TS^i\partial)$

$$= \sum_{i=0}^{m-1} (\partial TS^i + T\partial S^i)$$

$$= \sum_{i=0}^{m-1} (\partial T + T\partial) S^i$$

$$= \sum_{i=0}^{m-1} (1-S) S^i = \sum_{i=0}^{m-1} (S^i - S^{i+1}) = 1 - S^m.$$

For each singular n-simplex $\sigma: \Delta^n \to X$, there is an m such that $S^m(\sigma)$ lies in $C_n^{\mathcal{U}}(X)$, as $\sigma^{-1}(u_i)$ is an open cover of Δ^n, which has a Lebesgue number $\epsilon > 0$.

Define $m(\sigma) =$ smallest such m that works for σ.

we may now define : $D: C_n(X) \to C_{n+1}(X)$

$$\sigma \longmapsto D_{\sigma(m)}\sigma.$$

claim D is a chain homotopy

~~check~~: i.e. $\partial D_{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma = \sigma - S^{m(\sigma)}\sigma$

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Def^n $D : C_n(X) \to C_{n+1}(X)$

$$\sigma \longmapsto D_{m(\sigma)}\,\sigma$$

use: ~~want~~: chain homotopy equation for $D_{m(\sigma)}$:

~~note~~: $\partial D_{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma = \cancel{\sigma}\ \mathbb{1}\sigma - S^{m(\sigma)}\sigma$

$$(\ast) \quad \partial D\sigma + D\partial\sigma = \mathbb{1}\sigma - \left[S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) \right]$$

↑ define this to be $\rho(\sigma)$.

we get: $\partial D\sigma + D\partial\sigma = \sigma - \rho(\sigma)$.

<u>claim</u> $\rho(\sigma) \in C_n^{\mathcal{U}}(X)$

<u>check</u>: $S^{m(\sigma)}\sigma$ ✓

$D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$: if σ_j is j-th face of σ then $m(\sigma_j) \leq m(\sigma)$.

so this is a sum of terms $TS^i(\sigma_j)$ with $i \geq m(\sigma_j)$

and so all terms lie in $C_n^{\mathcal{U}}(X)$ ✓

so we have:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, shift right, "\rho"'] & C_n(X) \arrow[r] \arrow[d, shift right, "\rho"'] \arrow[dl, "D"'] & C_{n-1}(X) \arrow[r] \arrow[d, shift right, "\rho"'] \arrow[dl, "D"'] & \cdots \\
\cdots \arrow[r] & C_{n+1}^{\mathcal{U}}(X) \arrow[r] \arrow[u, shift right, "i"'] & C_n^{\mathcal{U}}(X) \arrow[r] \arrow[u, shift right, "i"'] & C_{n-1}^{\mathcal{U}}(X) \arrow[u, shift right, "i"'] &
\end{tikzcd}$$

and $(\ast)$ gives $\partial D + D\partial = \mathbb{1} - i\rho$, and $\rho i = \mathbb{1}$

so subdivision map chain homotopic to identity, so

$H_n(X) \cong H_n^{\mathcal{U}}(X)$ for all $n, \mathcal{U}$. $\square$.