choices • i injective, so a uniquely determined by ∂b.

• suppose we chose b' instead of b, note: $j(b') = j(b)$, so $b'-b$ $\in \ker(j) = \mathrm{im}(i)$, so $b'-b = i(a')$ for some $a' \in A_n$, so $b' = b + i(a')$

this replaces a by $a + \partial a'$, as
$$\begin{aligned} i(a+\partial a') &= i(a) + i\partial a' \\ &= ia + \partial i a' \\ &= \partial b + \partial i a' \\ &= \partial(b + ia') \end{aligned}$$
but $[a]$ and $[a+\partial a']$ define same homology class in $H_{n-1}(A)$.

• suppose we chose $c + \partial c'$ instead of c to represent $[c]$.

$c' = j(b')$ for some $b' \in B_n$, so $c + \partial c' = c + \partial j(b) = c + j(\partial b')$ $= j(b + \partial b')$, so b replaced by $b + \partial b'$, so doesn't change a.

check $\partial : H_n(C) \to H_{n-1}(A)$ is a homomorphism

suppose $\partial[c_1] = [a_1]$, $\partial[c_2] = [a_2]$ via b_1, b_2 as above,

then $j(b_1 + b_2) = j(b_1) + j(b_2) = c_1 + c_2$ and
$i(a_1 + a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1 + b_2)$

so $\partial([c_1] + [c_2]) = [a_1] + [a_2]$.

~~conclusions~~ finally

Thus $\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$ is exact $\square$.

Proof (left to show exactness).

$\mathrm{im}(i_*) = \ker(j_*)$ follows as $ji = 0 \Rightarrow j_* i_* = 0$

$\mathrm{im}(j_*) \subset \ker(\partial)$ $\quad \partial j_* = 0$ as $\partial b = 0$ in defn of ∂.

$\mathrm{im}(\partial) \subset \ker(i_*)$ $\quad i_*\partial = 0$ as $i_*\partial : [c] \mapsto [\partial b] = 0$.

$\ker(j_*) \subset \ker(\partial)$: let $[b] \in \ker(j_*)$, $b \in B_n$, so $j(b) = \partial c$ for some $c \in C_{n+1}$. j_* surjective, so $\exists\, b' \in B_{n+1}$ s.t. $j_*(b') = c$

$$\begin{tikzcd}
 & & A_n \arrow[d, "i_*"] & a \arrow[d, mapsto] \\
b' \arrow[ddd, mapsto] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j_*"] & B_n \arrow[d, "j_*"] & b - \partial b' \arrow[dd, mapsto] \\
 & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[d] & \\
 & 0 & 0 = [0] = j_*(b) & \\
c \arrow[r, mapsto] & \partial c = j(b) & &
\end{tikzcd}$$

note: $j(b - \partial b') = j(b) - j\partial b' = j(b) - \partial j b' \in 0$
$\quad = j(b) - \partial c = 0$

so $b - \partial b' = i(a)$ for some $a \in A_n$

<u>claim</u>: a is a cycle: $i(\partial a) = \partial i(a) = \partial(b - \partial b')$
$\quad = \partial b - 0 = 0$ ✓.
injective.

so $i_*[a] = [b - \partial b'] = [b]$, so i_* maps onto $\ker j_*$.

$\ker(\partial) \subset \operatorname{im}(j_*)$: let $[c] \in \ker(\partial)$, and let $[a] = \partial[c]$

$[a] = 0 \Rightarrow a = \partial a'$ for some $a' \in A_n$. Then $b - i(a')$ is a cycle as
$\partial(b - i(a')) = \partial b - \partial i(a') = \partial b - i(\partial a') = \partial b - i(a) = 0$.
note: $j(b - i(a')) = j(b) - ji(a') = j(b) = c$, so j_* maps $[b - i(a')]$ to $[c]$.

$\ker(i_*) \subset \operatorname{im}(\partial)$: let $a \in A_{n-1}$ be a cycle s.t. $i(a) = \partial b$ for some $b \in B_n$. Then $j(b)$ is a cycle as $\partial j(b) = j(\partial b) = ji(a) = 0$ and ∂ takes $[j(b)]$ to $[a]$.

So long exact sequence is exact $\square$.

Long exact sequence of the pair (X, A):

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

$\partial : H_n(X, A) \to H_{n-1}(A)$ this map is: let $[\alpha] \in H_n(X, A)$ so α is a relative cycle, then $\partial[\alpha]$ is the class of $\partial\alpha$ in $H_{n-1}(A)$.

Examples ① $H_k(D^n, \partial D^n) = \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{else} \end{cases}$ ② $x_0 \in X$

$H_n(X, x_0) = \tilde{H}_n(X)\ \forall n.$

Useful facts ① If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs, i.e. $\exists$ $F : (X \times I, A \times I) \to (Y, B)$ s.t. $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$, then $f_* = g_* : H_n(X, A) \to H_n(Y, B)$, $\forall n$. $\square$.

② there is a long exact sequence of a triple $B \subset A \subset X$ then we get $0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0$ short exact

gives $\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \cdots$ exact.

Excision Q: suppose $Z \subset A \subset X$. when does deleting Z leave $H_n(X, A)$ unchanged?

Thm (Excision 2.20) Let $Z \subset A \subset X$ s.t. closure of Z is contained in the interior of A, then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$ for all n.

Equivalently: if $A, B \subset X$ s.t. ~~$A \cup B$~~ $\text{int}(A) \cup \text{int}(B) = X$, then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \cong H(X, A)$ for all n.

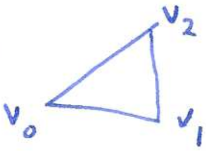
Proof uses barycentric subdivision. note: for a metric space $B_\epsilon(x)$ form an open cover, so gives notion of "small" subdivision. In Top, just use covers.

Def^n Let $\mathcal{U} = \{U_i\}$ be a collection of sets whose interiors form an open cover of X. $C_n^{\mathcal{U}}(X) \subset C_n(X)$, consists of all chains $\sum \sigma_i n_i$ in $C_n(X)$ s.t. for all σ_i, $\sigma_i(\Delta^n) \subset U_{j_i}$ for some set in $\mathcal{U}$.

Note: $\partial : C_n^{\mathcal{U}}(X) \to C_{n-1}^{\mathcal{U}}(X)$ so $C_n^{\mathcal{U}}(X)$ forms a chain complex.

Notation: homology groups are $H_n^{\mathcal{U}}(X)$.

<u>Prop^n</u>: The inclusion map $i: C_n^u(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence, i.e. there is a chain map $p: C_n(X) \to C_n^u(X)$ such that ip and pi are chain homotopic to the identity. This gives isomorphisms $H_n^u(X) \cong H_n(X)$ for all n.

<u>Proof</u> 1) <u>subdivide simplices</u>: 

$[v_0, \dots, v_n]$ has barycentric coordinates $\sum t_i v_i$, $\sum t_i = 1$, $t_i \geq 0$

the <u>barycenter</u> is $b = \sum t_i v_i$, with all t_i equal, i.e. $t_i = \frac{1}{n+1}$

the <u>barycentric subdivision</u> of $[v_0, v_1, \dots, v_n]$ is a decomposition of the simplex into simplices $[b, w_0, \dots, w_{n-1}]$ where $[w_0, \dots, w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivision of a face of Δ^n.

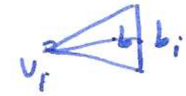
etc. <u>Fact</u>: this gives a Δ-complex structure on Δ^n.

<u>Fact</u>: diameter goes down: $\operatorname{diam}(\sigma') \leq \frac{n}{n+1} \operatorname{diam}(\sigma)$.
↑ in Euclidean metric on $\mathbb{R}^{n+1}$ ✓.

<u>warning</u>: if you just subdivide diam $\not\to 0$ e.g. 

<u>note</u> $\operatorname{diam}(\sigma) = $ max distance between any two vertices, as

$$|v - \sum t_i v_i| = |\textstyle\sum_i t_i (v - v_i)| \leq (\sum_i t_i) |v - v_i| \leq \max |v - v_i| .$$

now show bound is $\frac{n}{n+1}$: if neither w_i, w_j barycenter, then done by induction, so wma we have b, v_i. Let b_i be the barycenter of the opposite face $[v_0, \dots \hat{v}_i, \dots v_n] \leftarrow t_i = \frac{1}{n}$ in barycentric coords.


so $b = \frac{1}{n+1} v_i + \frac{n}{n+1} b_i$, and b lies on line segment from v_i to b_i, and $|b - v_i| \leq \frac{n}{n+1} |v_i - b_i|$, so $|b - v_i| \leq \frac{n}{n+1} \operatorname{diam}([v_0, \dots v_n])$.

<u>Corollary</u> in the r-th barycentric subdivision, diam of each simplex is at most $(\frac{n}{n+1})^r \operatorname{diam}([v_0 \dots v_n])$.

2) <u>subdivide linear chains</u>: construct $S: C_n(X) \to C_n(X)$
chain $\longmapsto$ subdivided chain