Def^n let $f_\#, g_\#$ be chain maps:

$$\begin{array}{ccccccc} \cdots \to & C_{n+1}(X) & \to & C_n(X) & \to & C_{n-1}(X) & \to \cdots \\ & \downarrow\downarrow & & f_\# \downarrow\downarrow g_\# & & \downarrow\downarrow & \\ \cdots \to & C_{n+1}(Y) & \to & C_n(Y) & \to & C_{n-1}(Y) & \to \cdots \end{array}$$

a <u>chain homotopy</u> is a collection of maps $P: C_n(X) \to C_{n+1}(Y)$

s.t. $\partial P + P \partial = g_\# - f_\#$

Prop^n Chain homotopic maps induce the same homomorphisms on homology $\square$.

<u>Fact</u>: this works for reduced homology. $\square$.

<u>Recall</u>: ~~we have~~ this shows homotopy equivalences give isomorphisms on homology groups.

—//—

Exact sequences and excision

common construction $A \subseteq X$, take quotient space X/A.

<u>Examples</u> $X = [0,1]$, $A = \{0,1\}$. $X/A = S^1$, Δ^n, $\partial\Delta^n \rightsquigarrow S^n$, Δ-complex...

<u>aim</u>: if we know $H_*(A)$, $H_*(X)$, can we find $H_*(X/A)$?

tool: <u>Def^n</u> (exact sequence) A sequence of abelian groups and homomorphisms between them as $\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$ is <u>exact</u> if

$\ker(\alpha_n) = \operatorname{im}(\alpha_{n+1})$ for all n. i.e. An exact sequence is a chain complex with trivial homology groups.

<u>Observations</u>

- $0 \to A \xrightarrow{\alpha} B$ exact $\Leftrightarrow$ α injective.
- $A \xrightarrow{\alpha} B \to 0$ exact $\Leftrightarrow$ α surjective.
- $0 \to A \xrightarrow{\alpha} B \to 0$ exact $\Leftrightarrow$ α isomorphism.
- $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ called a <u>short exact sequence</u>.

$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ (short) exact iff $\left.\begin{array}{l}\alpha \text{ injective} \\ \beta \text{ surjective} \\ \ker(\beta) = \operatorname{im}(\alpha)\end{array}\right\} \Rightarrow C \cong B/\operatorname{im}(\alpha)$ (abuse of notation, consider $A \subseteq B$). $C \cong B/A$.

excision/
long exact sequence of a pair

Thm (2.13, ~~excision~~) X space, $A \subseteq X$ non-empty, closed, deformation retract of an open neighbourhood in X, then there is an exact sequence:

$$\tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots$$

$$\cdots \to \tilde{H}_0(X/A) \to 0.$$

where i is the inclusion map $A \hookrightarrow X$
j is the quotient map $X \to X/A$.
∂ constructed in course of proof, idea:

Notation (X,A) denotes a (good) pair of spaces, $A \subseteq X$, $A \neq \emptyset$, A closed, and is a deformation retract of an open nbd of X.

Example X Δ-complex, $A \subseteq X$ subcomplex, (X,A) is a good pair.
CW-complex, sub CW-complex.

Corollary $\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_k(S^n) = 0$ for $k \neq n$.

Proof take $(X,A) = (D^n, S^{n-1})$ or $(\Delta^n, \partial\Delta^n)$, then $X/A \cong S^n$. $D^n \cong \Delta^n$ contractible, so every third term in long exact sequence of a pair is zero:

$$\tilde{H}_k(\partial\Delta^n) \xrightarrow{i_*} \underset{0}{\tilde{H}_k(\Delta^n)} \xrightarrow{j_*} \tilde{H}_k(S^n) \xrightarrow{\partial}$$

$$H_{k-1}(\partial\Delta^n) \to \underset{0}{H_{k-1}(\Delta^n)} \to H_{k-1}(S^n).$$

and $\partial\Delta^n \cong S^{n-1}$

so $\tilde{H}_k(S^n) \cong H_{k-1}(S^{n-1})$ and $H_k(S^0) = \begin{cases} \mathbb{Z} & k=0 \\ 0 & \text{else} \end{cases}$

result follows by induction. $\square$

Corollary (Brouwer fixed pt Thm) ∂D^n is not a retract of D^n, so every map $f: D^n \to D^n$ has a fixed point.

Proof: Suppose $r: D^n \to \partial D^n$ is a retraction, then $ri = \mathbb{1}$, for $i: \partial D^n \hookrightarrow D^n$ and $\mathbb{1}_{\partial D^n}$ so

$$\underset{\mathbb{Z}}{\tilde{H}_{n-1}(\partial D^n)} \xrightarrow{i_*} \underset{0}{H_{n-1}(D^n)} \xrightarrow{r_*} \underset{\mathbb{Z}}{H_{n-1}(\partial D^n)}$$

so $r_* i_* = 0 \neq \mathbb{1}_* = \mathbb{1}_{\mathbb{Z}}$ $\#$ $\square$.

Relative homology groups (gives version of long exact seq of pair).

(X, A) pair of spaces.

Let $C_n(X, A) = C_n(X) / C_n(A)$, i.e. chains in A trivial in $C_n(X,A)$.

note: $\partial: C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$.

so induces a quotient map $\partial: C_n(X,A) \to C_{n-1}(X,A)$

so gives a chain complex $\cdots \to C_{n+1}(X,A) \xrightarrow{\partial_{n+1}} C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A) \to \cdots$

as $\partial^2 = 0$.

Defn the homology groups of this chain complex are the relative homology groups $H_k(X,A)$.

Observations • elements of $H_n(X,A)$ are represented by relative cycles

i.e. an n-chain $\alpha \in C_n(X)$ s.t. $\partial\alpha \in C_{n-1}(A)$

• a relative cycle is ~~trivial~~ zero if it is a relative boundary, i.e.

$\alpha = \partial\beta + \gamma$, $\beta \in C_{n+1}(X)$, $\gamma \in C_n(A)$.

aim: construct long exact sequence:

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$

$$\cdots \to H_0(X,A) \to 0$$

this is an algebraic construction: we have short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "\partial"] & C_n(X) \arrow[r, "j"] \arrow[d, "\partial"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i"] \arrow[d] & C_{n-1}(X) \arrow[r, "j"] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

i = inclusion
j = quotient

check: this diagram commutes!

this is a short exact sequence of chain complexes $\rightsquigarrow$ long exact sequence.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r] \arrow[d] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
& 0 & 0 & 0 &
\end{tikzcd}$$

- diagram commutes $\Rightarrow$ i, j are chain maps and so give induced maps
  $i_* : H_n(A) \to H_n(B)$
  $j_* : H_n(B) \to H_n(C)$.

gives: $\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$

define: $\partial : H_n(C) \mapsto H_{n-1}(A)$

let $c \in C_n$, w/ $\partial c = 0$, j onto, so $\exists$ $b \in B_n$ s.t. $jb = c$

$$\begin{tikzcd}
& a \in A_{n-1} \arrow[d, mapsto, "i"] & A_{n-1} \arrow[d, "i"] \\
b \arrow[r, mapsto, "\partial"] \arrow[d, mapsto] & \partial b & \\
b \in B_n \arrow[rr, "\partial"] \arrow[d, "j"] & & B_{n-1} \arrow[d, "j"] \\
c \in C_n \arrow[rr, "\partial"] & & C_{n-1}
\end{tikzcd}$$

note: ~~$\partial b = 0 \neq \partial jb$~~
$\partial jb = \partial c = 0 = j \partial b$
so $\partial b \in \ker(j) = \operatorname{im}(i)$
$\Rightarrow$ $\exists$ $a \in A_{n-1}$ s.t. $i(a) = \partial b$

note: $i(\partial a) = \partial(ia) = \partial\partial b = 0$
i injective $\Rightarrow$ $\partial a = 0$.
So a defines a homology class $[a] \in H_{n-1}(A)$.

Define $\partial : [c] \mapsto [a]$. check: well defined!