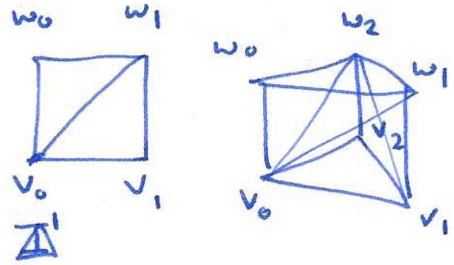
E.g.:



label $\Delta^n \times \{1\}$ $[w_0, w_1, \dots w_n]$

$\Delta^n \times \{0\}$ $[v_0, v_1, \dots v_n]$

intermediate n-simplices are

$[v_0, \dots, v_i, w_{i+1}, \dots w_n]$.

<u>claim</u>: this works, i.e. divides $\Delta^n \times I$ into $(n+1)$-simplices:

$[v_0, \dots, v_i, w_i, w_{i+1}, \dots, w_n]$.

observe: ① $\Delta^n \times I$ has coordinates $(\underset{\nearrow}{t_i}, s)$

barycentric coords on Δ^n

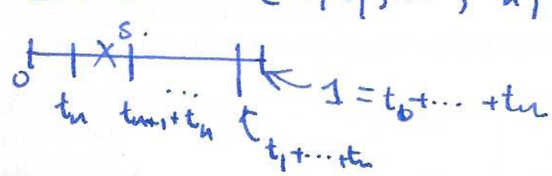
so intermediate n-simplices are given by $\phi_i : (t_i) \mapsto t_0 v_0 + \dots + t_i v_i + t_{i+1} w_{i+1} + \dots + t_n w_n$

$= (t_0, t_1, \dots, t_n, \underbrace{t_{i+1} + t_{i+2} + \dots t_n}_{s \text{ coord}})$. For points in interior of Δ^n, each

$t_i > 0$, so interiors of $\phi_i(\Delta^n)$ all disjoint, and $\phi_i(\mathring{\Delta}^n) < \phi_{i+1}(\mathring{\Delta}^n)$.

② region between $\phi_i(\Delta^n)$ and $\phi_{i+1}(\Delta^n)$ is the $(n+1)$-simplex with vertices $[v_0, v_1, \dots, v_i, w_i, w_{i+1}, \dots, w_n]$.

③ each point in interior $\Delta^n \times I \Rightarrow (t_0, t_1, \dots, t_n, s)$ lies in same $(n+1)$-simplex, which are:



<u>Conclusion</u> the simplices: $[v_0, v_1 \dots v_i, w_i, \dots w_n]$ form a Δ-complex structure on $\Delta^n \times I$.

—//—

Let $F : X \times I \to Y$ be a homotopy from f to g

want

define the <u>prism operator</u> $P : C_n(X) \to C_{n+1}(Y)$ by

$$P(\sigma) = \sum_i (-1)^i \underbrace{F \circ (\sigma \times \mathbb{1})}_{} \Big|_{[v_0, v_1, \dots, v_i, w_i, w_{i+1}, \dots, w_n]}.$$

signs take care of orientations.

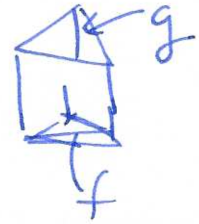
$\Delta^n \times I \xrightarrow{\sigma \times \mathbb{1}} X \times I \xrightarrow{F} Y$.

at chain level:

$$\begin{tikzcd} C_{n+1}(X) \arrow[r, "\partial"] & C_n(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] & C_{n-1}(X) \arrow[dl, "P"] \\ C_{n+1}(Y) \arrow[r, "\partial"'] & C_n(Y) \arrow[r, "\partial"'] & C_{n-1}(Y) \end{tikzcd}$$

<u>claim</u> $\boxed{\partial P = g_\# - f_\# - P\partial}$ ← sides! $(\partial\Delta^n)\times I$

boundary of the prism $\Delta^n \times I$. $\quad \Delta^n\times\{1\}$ top $\quad \Delta^n \times \{0\}$ bottom



sides.

$(\partial\Delta^n)\times I$

<u>Proof</u> (of claim) $\quad P(\sigma) = \sum_{i=0}^{n} (-1)^i F\circ(\sigma\times\mathbb{1})\big|[v_0,\dots,v_i,w_i,\dots w_n]$.

$$\partial P(\sigma) = \sum_{j\leq i} (-1)^i(-1)^j F\circ(\sigma\times\mathbb{1})\big|[v_0,\dots \hat{v}_j,\dots v_i,w_i,\dots,w_n].$$
$$+ \sum_{j\geq i} (-1)^i(-1)^{j+1} F\circ(\sigma\times\mathbb{1})\big|[v_0,\dots,v_i,w_i,\dots \hat{w}_j,\dots,w_n]$$

- terms with $i=j$ all cancel out, except first and last:

$$F\circ(\sigma\times\mathbb{1})\big|[\hat{v}_0,w_0,\dots,w_n] = \cancel{f}\, g\circ\sigma = \cancel{f_\#}\, g_\#(\sigma).$$

and $-F\circ(\sigma\times\mathbb{1})\big|[v_0,v_1,\dots,v_n,\hat{w}_n] = -f\circ\sigma = -f_\#(\sigma)$.

- terms with $i\neq j$ are exactly $-P\partial(\sigma)$, as $\quad \partial\sigma = \sum_{j=0}^{n}(-1)^j[v_0,\dots,\hat{v}_j,\dots,v_n]$

$$P\partial(\sigma) = \sum_{i<j}(-1)^i(-1)^j F\circ(\sigma\times\mathbb{1})\big|[v_0,\dots v_i,w_i,\dots \hat{w}_j \dots w_n].$$
$$+ \sum_{j>i}(-1)^{i-1}(-1)^j F\circ(\sigma\times\mathbb{1})\big|[v_0,\dots \hat{v}_j,\dots v_i,w_i,\dots,w_n]. \quad \square$$

Final bit: let $\alpha\in C_n(X)$ be a cycle, so $\partial\alpha = 0$, then.

$$\partial P(\alpha) = g_\#(\alpha) - f_\#(\alpha) - \underbrace{P\partial(\alpha)}_{=0} \Rightarrow g_\#(\alpha) = f_\#(\alpha) + \partial P(\alpha),$$

so same homology class, so $f_* = g_*$ on H_k. $\square$.