Proof $\sigma : \Delta^n \to X$ has connected image, as Δ^n connected, so can partition bases for $C_n(X)$ by which connected component $\sigma(\Delta^n)$ lives in, so $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$. Note that $\partial_n \sigma_n$ consists of $(n-1)$simplices lying in same connected component, so ∂_n preserves this connect sum decomposition. Fact: direct sum of chain complexes gives direct sum of homology groups. $\square$

Prop$^{\underline{n}}$ If X is non-empty, path connected, then $H_0(X) \cong \mathbb{Z}$. So for any X, $H_0(X) \cong \mathbb{Z}^{\#\text{path components}}$.

Proof $\cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0$ so $H_0(X) = C_0(X) / \mathrm{im}(\partial_1)$.

X

let $\sigma_1 : [v_0] \to X$ then as X path connected, $\exists \gamma : I \to X$
$\sigma_2 \; [v_1] \to X$ s.t. $\gamma(0) = \sigma_1(v_0)$, $\gamma(1) = \sigma_2(v_1)$.

note: $\gamma \in C_1(X)$, basis element, and $\partial\gamma = \sigma_2(v_0) - \sigma_1(v_1)$

more formally: define $\epsilon : C_0(X) \to \mathbb{Z}$ $(\sum n_i \sigma_i) \mapsto (\sum_i n_i)$

note: if X non-empty, then ϵ surjective.

claim: $\ker(\epsilon) = \mathrm{im}(\partial_1)$ if X path connected

note $\epsilon \partial_1(\sigma) = \epsilon(\sigma|_{[v_1]} - \sigma|_{[v_0]}) = 1 - 1 = 0$.

so $\mathrm{im}(\partial_1) \subseteq \ker(\epsilon)$.

show $\ker(\epsilon) \subseteq \mathrm{im}(\partial_1)$: suppose $\epsilon(\sum n_i \sigma_i) = 0 \Rightarrow \sum_i n_i = 0$

pick basepoint $x \in X$, and for each σ_i choose a path γ_i from x to $\sigma_i(v_0)$, so $\gamma_i \in C_1(X)$ and $\partial\gamma_i = \sigma_i - x$

so $\partial(\sum n_i \gamma_i) = \sum n_i \sigma_i - \underbrace{\sum_i n_i}_{0} x = \sum n_i \sigma_i$ as required $\square$.

Prop$^{\underline{n}}$ If $X = \{pt\}$ then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \cong \mathbb{Z}$

Proof note there is a unique map $\sigma : \Delta^n \to X$.

so we get

$$\cdots \to C_2(X) \to C_1(X) \to C_0(X) \to 0$$
$$\cdots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$$

$$\partial(\sigma_n) = \sum_i (-1)^i \sigma_{n-1} \quad \leftarrow n+1 \text{ terms} \quad = \begin{cases} 0 & n \text{ even} \\ \sigma_{n-1} & n \text{ odd} \end{cases}$$

so the maps are:

$$\cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$$\cdots \quad H_2 = 0 \quad H_1 = 0 \quad H_0(\cancel{X}) \cong \mathbb{Z} \quad \square.$$

Def^n Reduced homology (add an coefficient map at end).

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{0} 0 \qquad (X \neq \emptyset)$$

can think of this as having basis $[\emptyset]$.

notation reduced homology grp $\tilde{H}_n(X)$.

observation: $H_n(X) \cong \tilde{H}_n(X)$ if $n \geq 1$

$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$ $n = 0$.

Example $\tilde{H}_n(\{pt\}) = 0$ for all n.

Induced maps $f: X \to Y$ cts, give $f_*: H_n(X) \to H_n(Y)$ homomorphism.

furthermore, ~~if f is a homotopy equivalence, then f_* is an isomorphism.~~ homotopic maps give the same homomorphism, which implies that if f is a homotopy equivalence, the f_* is an isomorphism.

note if ~~$\oint$~~ $\sigma: \Delta^n \to X$ and $f: X \to Y$ then $f\sigma: \Delta^n \to Y$

define: $f_\#: C_n(X) \to C_n(Y)$

$\sigma \longmapsto f\sigma$, defined on basis elements, and extend linearly this gives maps:

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

claim: this diagram commutes, i.e. $f_\# \partial = \partial f_\#$

check: $f_\# \partial(\sigma) = f_\#\left(\sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots v_n]} \right)$

$$= \sum_i (-1)^i f\sigma|_{[v_0, \dots, \hat{v}_i, \dots v_n]} = \partial f_\# \sigma \quad \square$$

Def$^{\underline{n}}$ A diagram in which all the compositions of maps commute is called a commutative diagram.

Def$^{\underline{n}}$ Let C_n, D_n be chain complexes. A collection of maps $f_n : C_n \to D_n$ forming a commutative diagram is called a chain map.

Observations
- $f_\#$ takes cycles to cycles: if $\partial\alpha = 0$ then $\partial f\alpha = f\partial\alpha = f0 = 0$
- $f_\#$ takes boundaries to boundaries: if $\alpha = \partial\beta$ then $f\alpha = f\partial\beta = \partial f\beta$.

this implies:

Prop$^{\underline{n}}$ A chain map between chain complexes induces homomorphisms between their homology groups.

notation: $f_\#: C_n(X) \to C_n(Y)$ $\qquad$ $f_*: H_n(X) \to H_n(Y)$

Useful properties

① $(fg)_* = f_* g_*$ for compositions $\Delta^n \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z$

(composition of maps is associative)

② $\mathbb{1}_* = \mathbb{1}$ identity map $X \xrightarrow{\mathbb{1}} X$ induces identity isomorphism $\mathbb{1}_*: H_n(X) \to H_n(X)$.

Thm 2.10 If two maps $f, g: X \to Y$ are homotopic, they induce the same homomorphism $f_* = g_*: H_n(X) \to H_n(Y)$

Corollary If $f: X \to Y$ is a homotopy equivalence, then f_* is an isomorphism.

Proof (of corollary) $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ $\qquad$ $gf \simeq \mathbb{1}_X$, $fg \simeq \mathbb{1}_Y$

so

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\mathbb{1}_X"'] & Y \arrow[r, "g"] & X \end{tikzcd} \Rightarrow g_* f_* = \mathbb{1}_{X*}: H_n(X) \to H_n(X)$$

$$Y \xrightarrow{g} X \xrightarrow{f} Y \Rightarrow f_* g_* = \mathbb{1}: H_n(Y) \to H_n(Y)$$

$\Rightarrow f_*, g_*$ both surjective and injective, so isomorphisms $\square$.

Example If X contractible, then $H_n(X) = \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{if } n \geq 1. \end{cases}$

Proof (of 2.10: homotopic maps give same homomorphism on homology)

observation: we can triangulate $\Delta^n \times I$ as follows