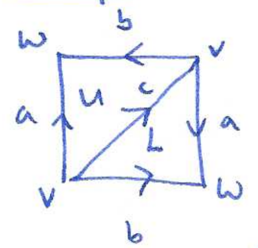
Example $X = \mathbb{RP}^2$, as before $\cdots \to C_3 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$

$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z}^2 \to 0$$

$\{u, L\}$ $\{a, b, c\}$




$u \mapsto -a+b+c$ $\quad$ $a \mapsto -v+w$

$L \mapsto -a+b-c$ $\quad$ $b \mapsto -v+w$

$c \mapsto 0$

check: $\underset{\partial_1}{\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}} \underset{\partial_2}{\begin{bmatrix} -1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad$ $\underset{\partial_2}{\begin{bmatrix} -1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}}$ $\quad$ $\underset{\partial_1}{\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}}$

change basis in $C_0 \leftrightarrow$ row ops in $\partial_1$: $\underset{\partial_1}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$ $\underset{\partial_2}{\begin{bmatrix} -1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}}$.

change basis in $C_1 \leftrightarrow$ col ops in $\partial_1$
row ops in $\partial_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ '' $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ '' $\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$

subtract col I from col 2 $\qquad$ add row 2 to row 1

$\underset{}{\overset{\partial_1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}} \overset{\partial_2}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}}$

change basis in $C_2 \leftrightarrow$ col ops in $\partial_2$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$.

ker dim 2 $\qquad$ image of $\partial_2$

So $H_1 = \mathbb{Z}^2 / \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.

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Q: does $H_k^\Delta(X)$ depend on $\Delta$-structure? A: No

① simplicial approximation (sketch) Thm $K$ finite simplicial complex and $L$ any simplicial complex, then any map $f: K \to L$ is homotopic to a simplicial map on some subdivision of $K$.

Extra steps define subdivision $\Delta \to$ show preserves $H_*$, check homotopic maps give same induced map on $H_*$.

② use functor defn of ~~chain complex~~ homology

## Singular homology

**Defn** A singular n-simplex is a cts map $\sigma : \Delta^n \to X$

The singular chain group $C_n(X)$ is the free abelian group with basis set of singular n-simplices of X. Elements of $C_n(X)$ are n-chains or finite formal sums $\sum_i n_i \sigma_i$, $n_i \in \mathbb{Z}$, $\sigma_i : \Delta^n \to X$.

The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma |[v_0, \dots, \hat{v}_i, \dots, v_n]$$

**Propn** $\partial_n \partial_{n+1} = 0$ or $\partial^2 = 0$ $\square$.

**Corollary** $\cdots \to C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \cdots$ is a chain complex

**Defn** the singular homology groups are $H_n(X) = \ker(\partial_n) / \operatorname{im}(\partial_{n+1})$

Observation: homeomorphic spaces have isomorphic singular homology groups.

Q: how to compute $H_n(X)$? $C_n(X)$ usually uncountable dimensional.

Observation: cycles in low dimensions: suppose $\partial(\sum n_i \sigma_i) = 0$.

cycle, i.e. chain $\alpha$ w $\partial\alpha = 0$.

can always write $\sum n_i \sigma_i$ as $\sum \epsilon_i \sigma_i$ w/ $\epsilon_i = \pm 1$

$\partial \sum \epsilon_i \sigma_i = \sum \epsilon_i (\partial \sigma_i) = 0$ ← sum of $(n-1)$-simplices

so a union of cancelling pairs i.e. each $(n-1)$-face occurs twice w/ opposite signs.

$H_1(X)$: cycles can be oriented loops

$H_2(X)$: cycles can be represented by maps of closed oriented surfaces.

$H_3(X)$: fail ∵ doesn't need to be 3-manifold as no restriction on $(n-2)$-d simplices.

Useful facts

**Propn** Let X have path components $X_\alpha$, then $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$.