Example ① $X = S^1$  i.e. $[v_0, v_1] / v_0 \approx v_1$.

$\Delta_0(S^1) \cong \mathbb{Z} = \langle v \rangle$

$\Delta_1(S^1) \cong \mathbb{Z} = \langle e \rangle$.

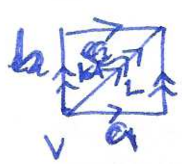
$$0 \to \Delta_1(S^1) \xrightarrow{\partial_1} \Delta_0(S^1) \to 0$$

$$0 \to \mathbb{Z} \xrightarrow[0]{} \mathbb{Z} \to 0$$

$$\langle e \rangle \longmapsto 0$$

$$\partial_1(e) = \partial_1([v_0, v_1]) = [v_1] - [v_0] = [v] - [v] = 0$$

so $H_1^\Delta(S^1) \cong \mathbb{Z}$ $\quad H_0^\Delta(S^1) \cong \mathbb{Z}$. $\quad H_k^\Delta(S^1) = 0$ for $k \geq 2$.

② $X = T^2$: 

$$0 \to \Delta_2(T^2) \xrightarrow{\partial_2} \Delta_1(T^2) \xrightarrow{\partial_1} \Delta_0(T^2) \xrightarrow{\partial_0} 0$$

$$\mathbb{Z}^2 \qquad \mathbb{Z}^3 \qquad \mathbb{Z}$$

$$\{u, L\} \qquad \{a, b, c\} \qquad \{v\}.$$

$\partial_2(u) = c - a - b$ $\qquad \partial_1(a) = v - v = 0$

$\partial_2(L) = a + b - c$ $\qquad \partial_1(b) = 0$

$\qquad \partial_1(c) = 0$ $\qquad$ so $\partial_1 = 0$.

so $\ker(\partial_2) \cong \mathbb{Z} = \langle u - L \rangle$.

$\quad \operatorname{im}(\partial_2) \cong \mathbb{Z} = \langle a + b - c \rangle$.

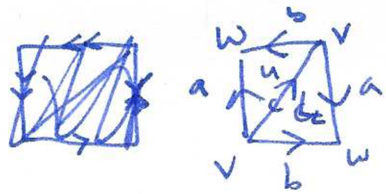
$$H_0^\Delta(T^2) = \ker(\partial_0) / \operatorname{im}(\partial_1) = \mathbb{Z}/0 = \mathbb{Z}.$$

$$H_1^\Delta(T^2) = \ker(\partial_1) / \operatorname{im}(\partial_2) = \{a, b, c\} / \{a+b-c\}.$$

<u>note</u>: $\{a, b, a+b-c\}$ is a basis for $\mathbb{Z}^3$.

$$= \{a, b, a+b-c\} / \{a+b-c\} \cong \mathbb{Z}^2.$$

$$H_2^\Delta(T^2) = \ker(\partial_2) / \operatorname{im}(\partial_3) = \mathbb{Z}/0 = \mathbb{Z}.$$

③ $\mathbb{RP}^2$: 

$$0 \to \Delta_2(\mathbb{RP}^2) \xrightarrow{\partial} \Delta_1(\mathbb{RP}^2) \xrightarrow{\partial} \Delta_0(\mathbb{RP}^2) \to 0$$

$$\mathbb{Z}^2 \qquad \mathbb{Z}^3 \qquad \mathbb{Z}^2$$

$$\{u, L\} \qquad \{a, b, c\} \qquad \{v, w\}.$$

$$u \longmapsto c-a+b \qquad a \longmapsto w-v$$
$$L \longmapsto b-a-c \qquad b \longmapsto w-v$$
$$c \longmapsto 0$$

$$H_0^\Delta(\mathbb{RP}^2) = \ker(\partial_0)/\operatorname{im}(\partial_1) = \{v,w\}/\{w-v\} = \{v, w-v\}/\{w-v\} \cong \mathbb{Z}.$$

$$H_1^\Delta(\mathbb{RP}^2) = \ker(\partial_1)/\operatorname{im}(\partial_2) = \{a-b, c\}/\{c-a+b, b-a-c\} = \{d, c\}/\{c-d, +d+c\}.$$

$$= \{c, d\}/\{d, 2c\} \cong \mathbb{Z}/2\mathbb{Z} \ (\text{or } \mathbb{Z}_2).$$

$$H_2^\Delta(\mathbb{RP}^2) = \ker(\partial_2)/\operatorname{im}(\partial_3) = \overset{0}{\cancel{\{u-L\}}}/0 \cong \cancel{\neq} 0.$$

so
$$H_k^\Delta(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/2\mathbb{Z} & k=1 \\ 0 & k \geq 2 \end{cases}$$

Q: • is $H_k^\Delta(X)$ independent of simplicial/Δ-complex structure on X? (yes)

• how to do effective calculations in abelian groups.

<u>Classification of finitely gen. abelian groups</u>

key point: $\mathbb{Z}/4\mathbb{Z} \not\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ but $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

$\mathbb{Z}_4 \not\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

<u>Thm</u> If G is a finitely generated abelian group then
$G \cong \mathbb{Z}^r \oplus \mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \oplus \cdots \oplus \mathbb{Z}_{q_k}$ where the q_i are not necessarily distinct prime powers.

~~warning: $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to 0$ $\qquad$ $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \to 0$~~

~~$H_* = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ $\qquad$ $H_\cdot = \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}_3$.~~

~~but $\mathbb{Z}^2 \to \mathbb{Z}^2 \to 0$, $(1,0) \mapsto (2,0)$, $(0,1) \mapsto (0,3)$ $\Big\}$ $H_\cdot = \mathbb{Z}$! as $\{(2,3),(1,1)\}$ is a basis for $\mathbb{Z}^2$.~~

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small examples: ad hoc methods work

large examples: Smith normal form.

setup: chain complex $\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$

want: $H_n = \ker(\partial_n)/\mathrm{im}(\partial_n)$

<u>note</u>: • given bases for C_n, ∂_n maps are matrices.

• $\ker(\mathbb{Z}^a \to \mathbb{Z}^b)$ always free abelian group, so just need to find $\mathrm{im}(\partial_n)$ inside it.

aim: change bases to make ∂_n diagonal.

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$$

$\{e_1, \ldots, e_k\}$ ← change of basis here gives <u>row operations</u> in $[\partial_n]$
<u>column operations</u> in $[\partial_{n+1}]$

<u>example</u>

$\{e_1, e_1+e_2, e_3, \ldots, e_k\}$ ← add row 1 to row 2 in $[\partial_n]$
<u>subtract</u> row 2 from row 1 in $[\partial_{n+1}]$.

<u>recall</u> $\partial_n \partial_{n+1} = 0$

$$[\partial_n][\partial_{n+1}] = 0.$$

$$[\partial_n]\underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{E_{12}^{-1}}\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{E_{12}}[\partial_{n+1}].$$

mult on right, col op. $\qquad$ mult on left, row op.