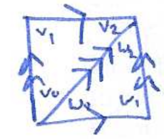
$f_i = [v_{0,i}, v_{1,i} \dots v_{k,i}]$ then $t \in f_i$ is $t = \sum_{j=0}^{k} t_j v_{j,i}$

and then identify $\sum t_j v_{j,i} \sim \sum t_j v_{j,l}$ for all i, l. $1 \leq i \leq l \leq k$

<u>Warning</u> Δ-complexes $\supseteq$ simplicial complexes. $\leftarrow$ every simplex homeomorphically embedded.

<u>Example</u>



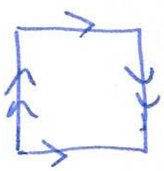
$\leftarrow$ this is a Δ-complex structure on T^2

it is <u>not</u> a simplicial complex structure on T^2.

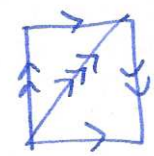
$[v_0, v_1, v_2]$ $[w_0, w_1, w_2]$ two disjoint 2-simplices.

identifications: $F = \{ \{[v_0, v_1], [w_1, w_2]\}, \{[v_1, v_2], [w_0, w_1]\}, \{[v_0, v_2], [w_0, w_2]\} \}$

<u>Example</u>

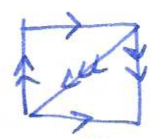


$\leftarrow$ Klein bottle. K



$\leftarrow$ this is a Δ-complex structure on K.

<u>warning</u>! :

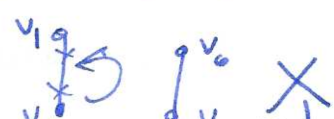


$\leftarrow$ this is <u>not</u> a Δ-complex structure on K!

<u>Facts</u> ① orientation compatibility condition: suffices to check no 2-simplex has a cyclic orientation on its boundary edges.

② for a 1-dimensional Δ-complex, no compatibility condition, so a 1-dim Δ-complex is exactly an oriented graph.

③ the identification / gluing maps preserve the ordering of the vertices, so no two points in the interior of a face are ever identified:

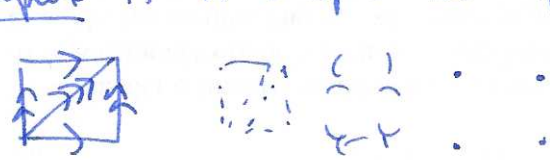


no!

④ so $X = \bigsqcup \Delta_\alpha / \sim$ is a disjoint union of <u>open</u> simplices.

<u>Defn</u> An <u>open simplex</u> is a simplex with its proper faces deleted.

<u>Examples</u> T^2.



In fact:

we call this map $\sigma_\alpha : \Delta^n_\alpha \to X$ the <u>characteristic map</u> for Δ^n_α, and it is a homeomorphism on the interior (not nec on ~~closed~~ full simplex!)

Def: Δ-complex / simplicial homology

X Δ-complex $\qquad \sigma_\alpha : \Delta^n \to X$.

$\Delta_n(X)$ free abelian group with basis open n-simplices of X, called chain groups

↑ formal sums of n-simplices $\sum_\alpha n_\alpha \sigma_\alpha$ ← an n-chain.

Example T^2: 

$\Delta_0(X) \cong \mathbb{Z} \qquad \Delta_1(X) \cong \mathbb{Z}^3 \qquad \Delta_2(X) \cong \mathbb{Z}^2$.

basis: $\{v\} \qquad \{e_1, e_2, e_3\} \qquad \{K, L\}$.

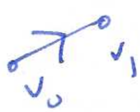
Def: Boundary of an n-simplex.

$$\partial\Delta^n = \partial[v_0, v_1, \dots, v_n] = \sum_i (-1)^i [v_0, v_1, \dots \hat{v}_i, \dots v_n].$$

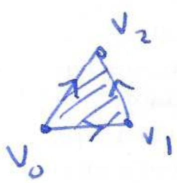
formal sum. $\quad$ sign takes care of orientation $\quad$ hat ^ means omit that letter.

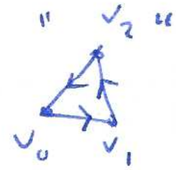
Examples

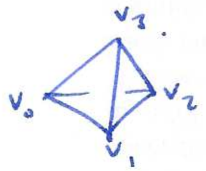
$[v_0]$. $\qquad \partial[v_0] = \emptyset$.

$[v_0, v_1]$ 

$\partial[v_0, v_1] = [v_1] - [v_0]$.

$[v_0, v_1, v_2]$ 

$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$ 

$[v_0, v_1, v_2, v_3]$ 

$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$.

Def: boundary map for chain complexes.

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X)$$

$$(\sigma_\alpha : \Delta^n \to X) \longmapsto \sum_{i=0}^{n} (-1)^i \sigma_\alpha|[v_0, \dots \hat{v}_i, \dots v_n].$$

suffices to define for basis elements and extend linearly

Lemma The composition of two boundary maps is zero.

i.e. $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ $\quad \partial_{n-1}\partial_n = 0$

Proof suffices to check for a basis element of $\Delta_n(X)$.

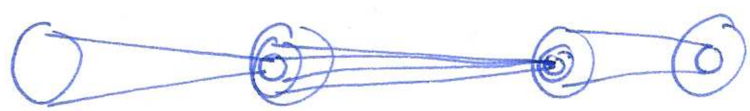
$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots v_n]}$$

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i} (-1)^i(-1)^j \sigma|_{[v_0,\dots \hat{v}_j, \dots \hat{v}_i \dots v_n]}.$$
$$+ \sum_{j>i} (-1)^i(-1)^{j-1} \sigma|_{[v_0,\dots \hat{v}_i \dots \hat{v}_j \dots v_n]} = 0 \quad \square.$$

Def (algebra) A chain complex is a sequence of abelian groups and homomorphisms $\dots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \dots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$
such that $\partial_{n-1}\partial_n = 0$ for all n.

Observation the simplicial chain groups and boundary maps form a chain complex $\dots \to \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \to \dots$

Fact $\partial_{n-1}\partial_n = 0 \Rightarrow \mathrm{im}(\partial_n) \subseteq \ker(\partial_{n-1})$



Def The n-th homology group of a chain group is $H_n = \ker(\partial_n) / \mathrm{im}(\partial_{n+1})$

For simplicial homology:

$$\Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \to \dots$$

we will write $\quad H_n^{\Delta}(X) = \ker(\partial_n) / \mathrm{im}(\partial_{n+1})$.