Topology I Math 70800 (second semester) 5417 ①

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Text: Algebraic Topology, Allen Hatcher.

HW: every two weeks. Final: take qual.

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§2 Homology Introduction/motivation

problem: classify topological spaces up to homeomorphism. ← too hard. e.g. all subsets of $\mathbb{R}^n$.

simpler problem: - look at "nice" topological spaces. is $\mathbb{R}^n \cong \mathbb{R}^m$ for $m \neq n$?

- find invariants to help us distinguish different spaces.

example: # of connected components. (path connected) Functor: $\text{Top} \to \mathbb{N}$

$X = \{pt\} \mapsto 1$

$X = \{pt\} \sqcup \{pt\} \mapsto 2$.

slightly fancier: $I =$ $I \setminus pt$  disconnected

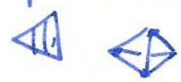
S^1  $S^1 \setminus pt \cong (0,1)$ connected.

Recall: fundamental group: $\text{Top} \to \text{Group}$. easy to define, hard to compute!

$X \mapsto \pi_1(X)$.

Examples $\pi_1(\mathbb{R}) \cong \{1\}$, $\pi_1(S^1) \cong \mathbb{Z}$, $\pi_1(T^2 \cong S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$ $\pi_1(S^1 \vee S^1) = F_2 = \langle a, b \,|\, \rangle$.

Problem: $X =$ simplicial complex. $\rightsquigarrow \pi_1(X) \leftarrow$ group presentation $\langle a_1, a_2, \ldots a_n \,|\, r_1, r_2, \ldots r_k \rangle$



Thm [Novikov] There is no algorithm to decide if a presentation gives the trivial group.

Higher dimensions: $\pi_k(X)$ $S^k \to X$, abelian groups, very hard to compute

$\pi_3(S^2) \cong \mathbb{Z}$. $\pi_k(S^n)$ unknown in general.

Homology : Top $\longrightarrow$ Abelian groups   more work to define, better tools for calculations.
$X \longmapsto H_1(X)$

- computable
- generalised to any dimension $H_k(X)$.

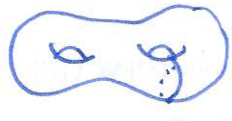
$H_0(X) = \mathbb{Z}^{\#\text{ path connected components}}$ $H_0(\mathbb{R}) = \mathbb{Z}$ $H_0(S^1)=\mathbb{Z}$ $H_0(T^2)$ $\mathbb{Z}$ $H_0(S^1\vee S^1)$ $\mathbb{Z}$

Fact $H_1(X) = ab(\pi_1(X))$, so $H_1(\mathbb{R})$ 0 $H_1(S^1)=\mathbb{Z}$ $H_1(T^2)$ $\mathbb{Z}\oplus\mathbb{Z}$ $H_1(S^1\vee S^1)$ $\mathbb{Z}\oplus\mathbb{Z}$.

intuition : $H_n(X) =$ "n-dim subsets without boundary" / "n-dim subsets which bound (n+1)-dim subsets".

= "closed sets" / "boundaries".

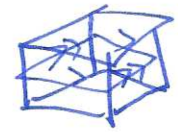
example :



$\neq 0 \in H_1(X)$

$= 0$ in $H_1(X)$.

$T^3 = S^1\times S^1 \times S^1$.



§2 Homology : simplicial homology

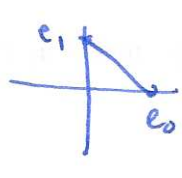
Def^n(s) An *n-simplex* is the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $v_0, v_1, \dots, v_n$ which ~~do not lie~~ are in general position, i.e. do not lie in an k-dim subspace for $k<n$. (equivalently, the vectors $v_1-v_0, v_2-v_0, \dots, v_n-v_0$ are linearly independent). The points v_i are the *vertices* of the simplex, which we shall denote $[v_0, v_1, \dots, v_n]$. For the purposes of defining homology groups, we will need to keep track of the order of the vertices, so this is an *ordered* list. Examples $[v_0]$. $[v_0,v_1]$

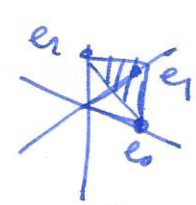
$[v_0,v_1,v_2]$

The *standard n-simplex* is $\Delta^n = \{ (t_0, t_1, \dots t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n+1} t_i = 1$ and $t_i \geq 0 \}$.
vertices are $[e_0, e_1, \dots, e_n]$ e_i standard basis vectors.

Examples : $[e_0]$

$[e_0, e_1]$





Remark all n-dimensional simplices are homeomorphic.

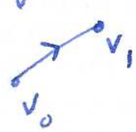
There is a canonical map from the standard n-simplex to any n-simplex given by

$$(t_0, t_1, \dots, t_n) \longmapsto t_0 v_0 + t_1 v_1 + \dots + t_n v_n$$

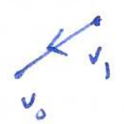
the coefficients t_i are called <u>barycentric coordinates</u> on $[v_0, v_1, \dots, v_n]$.

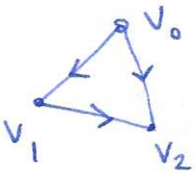
A <u>face</u> of an n-simplex is the k-simplex corresponding to a subset of $\{v_0, v_1, \dots v_n\}$. We include the full n-simplex as a face.

<u>convention</u>: we will always give the vertices of a face the order arising from the original simplex.

<u>Examples</u> $[v_0, v_1]$  faces $[v_0], [v_1], [v_0, v_1]$.

$\neq$

$[v_1, v_0]$ 

$[v_0, v_1, v_2]$  faces $[v_0], [v_1], [v_2]$

$[v_0, v_1]\ [v_1, v_2]\ [v_0, v_3]$

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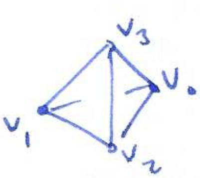
$[v_0, v_1, v_2, v_3]$ 

$[v_0], [v_1], [v_2], [v_3]$

$[v_0, v_1]\ [v_0, v_2]\ [v_0, v_3]\ [v_1, v_2]\ [v_1, v_3]\ [v_2, v_3]$

$[v_0, v_1, v_2]\ [v_0, v_2, v_3]\ [v_0, v_1, v_3]\ [v_1, v_2, v_3]$

$[v_0, v_1, v_2, v_3]$

etc.

<u>Def$^{\underline{n}}$</u> <u>Δ-complex</u>: a quotient of a collection of disjoint simplices obtained by identifying a some ~~collection~~ of their faces by the canonical linear homeomorphisms between them.

More formally: let Δ^n_α be a collection of disjoint simplices.

let F_β be a collection of sets of faces of the Δ^n_α, where each set consist of simplices of the same dimension.

$X =$ take $\bigsqcup \Delta^n_\alpha / \sim$ where $\sim$: if $F_i = (f_1, f_2, \dots f_k) \in F_\beta$ ~~then~~ and