Last time

$\pi_1(X, x_0)$ fundamental group

Prop$^{\underline{n}}$ $f : X \longrightarrow Y$ cts
$x_0 \longmapsto y_0$

gives homomorphism

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

Proof define $f_*[\gamma] = [f \circ \gamma]$

- well defined $[\gamma_1] = [\gamma_2]$ iff there is a homotopy $F : I \times I \longrightarrow X$

✓

- homomorphism

$$f_*[\gamma_1] \cdot f_*[\gamma_2] = f_*[\gamma_1 \cdot \gamma_2]$$

$$(f \circ \gamma_1) \cdot (f \circ \gamma_2) = f \circ (\gamma_1 \cdot \gamma_2) \checkmark \qquad \square$$

X Δ-complex $\rightsquigarrow$ presentation for $\pi_1(X, x_0)$

Group presentation:

$\langle g_1, \dots, g_k, | r_1, \dots, r_\ell \rangle$

words in g_i

Examples

$F_2 = \langle a, b | \rangle$

$\mathbb{Z}^2 = \langle a, b | ab = ba \rangle$

$\Updownarrow$ or $aba^{-1}b^{-1}$

$\mathbb{Z}_2 = \langle a | a^2 = 1 \rangle$

More formally let F = free group generated by $g_1, \dots, g_k$ take quotient F/N

N = normal subgroup

$gN = Ng$ for all $g \in G$

$(gN)(hN) = (gh)N$

generated by r_i

$\langle w r_i w^{-1} \rangle \ \forall w \in F$

Warning

$G = \langle g_1, \dots, g_k \mid r_1, \dots, r_\ell \rangle$

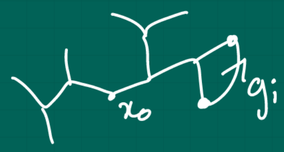
↑ no algorithm to tell if $G = \{1\}$

X Δ-complex path connected

x_0 vertex

T maximal tree in 1-skeleton $X^{(1)}$ ← 0 and 1-simplices

SKELETON



T

note: contains all the vertices

choose a generator g_i for each edge in $X^{(1)} - T$

tree T ← unique path connecting any point to x_0

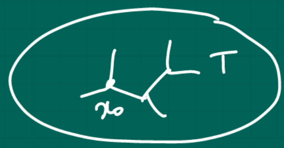
path in T from x_0 to end of $g_i(1)$ → γ_2

γ_1 ← path in T from x_0 to base of $g_i(0)$

$[\gamma_1 \cdot g_i \cdot \gamma_2] \leftrightarrow g_i$

2-simplices in X

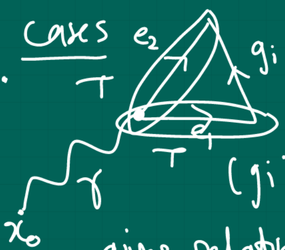
$\Longleftrightarrow$ relations



$\leftarrow$ can't have all edges in T

<u>cases</u>

• 2 edges in T



$(g_i) \leftrightarrow [\gamma \cdot e_1 \cdot g_i \cdot \bar{e}_2 \bar{\gamma}]$

$\uparrow$ homotopic to constant path

gives relation $g_i = 1$

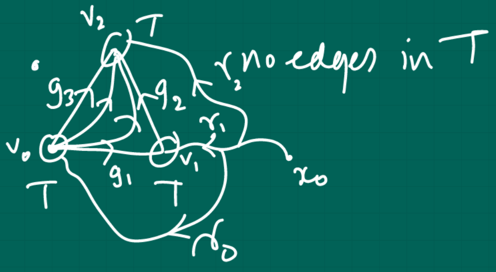
$\langle g_1, \dots, g_k \mid g_i = 1 \rangle$

• one edge in T

$g_j \leftrightarrow [\gamma_1 e_{12} g_j \gamma_2]$ } homotopic across triangle

$g_i \leftrightarrow [\gamma_1 g_i \gamma_2]$

relation $g_i = g_j$



relation: $g_1 g_2 = g_3$

check

$g_1 \Leftrightarrow [\gamma_0 g_1 \bar{\gamma}_1]$

$g_2 \Leftrightarrow [\gamma_1 g_2 \bar{\gamma}_2]$

$g_3 \Leftrightarrow [\gamma_0 g_3 \bar{\gamma}_2]$

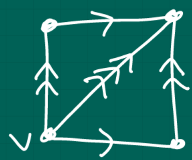
$$g_1 g_2 = [\gamma_0 g_1 \bar{\gamma}_1][\gamma_1 g_2 \bar{\gamma}_2]$$

$$= [\gamma_0 g_1 \underbrace{\bar{\gamma}_1 \gamma_1}_{\text{homotopic to constant}} g_2 \bar{\gamma}_2]$$

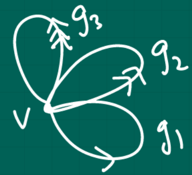
$$= [\gamma_0 \underbrace{g_1 g_2}_{\text{homotopic to } g_3} \bar{\gamma}_2]$$

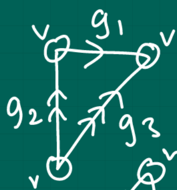
$$= [\gamma_0 g_3 \bar{\gamma}_2] = g_3$$

Example $\qquad T = \{v\}$

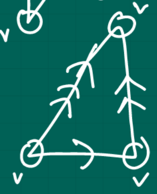


$X^{(1)}$





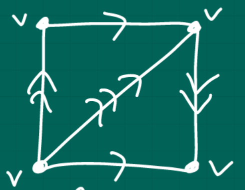
$\longleftrightarrow \quad g_2 g_1 = g_3$



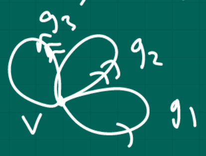
$\longleftrightarrow \quad g_1 g_2 = g_3$

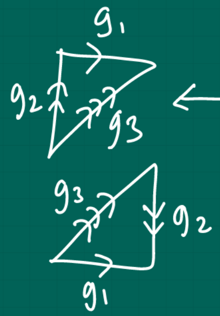
$$\langle g_1, g_2, g_3 \mid g_2 g_1 = g_3,\ g_1 g_2 = g_3 \rangle = \langle g_1, g_2 \mid g_1 g_2 = g_2 g_1 \rangle \cong \mathbb{Z}^2$$

Klein bottle



$X^{(1)}$





$\longleftrightarrow \quad g_2 g_1 = g_3$

$g_3 g_2 = g_1$

$$g_2^{-1} g_2 g_1 g_2 = g_2^{-1} g_1$$

$$g_1 g_2 = g_2^{-1} g_1$$

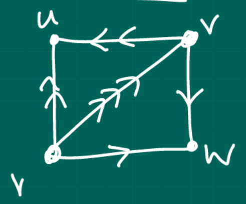
Fact/exerise

$\rightarrow$ not abelian

$$\langle g_1, g_2, g_3 \mid g_2 g_1 = g_3,\ g_3 g_2 = g_1 \rangle = \langle g_1, g_2 \mid g_2 g_1 g_2 = g_1 \rangle$$

Example

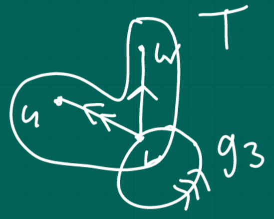
S^2



$\chi = V - E + F$

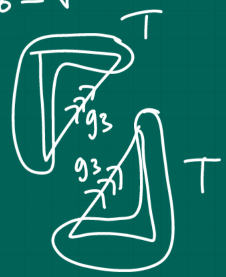
$3 - 3 + 2 = 2$ ✓

$X^{(1)}$



choose basepoint

$x_0 = v$



$g_3 = 1$

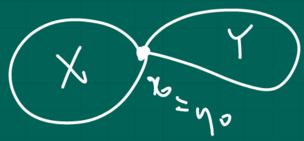
$g_3 = 1$

$\langle g_3 \mid g_3 = 1 \rangle =$ trivial group $= \{1\} = 1$

Corollary (simple) van Kampen Theorem

(X, x_0), (Y, y_0) Δ-complexes, path connected

$X \vee Y \leftarrow$ glue x_0 to y_0



then

$$\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$$

free product

$$\langle a_1, \dots, a_k \mid r_1, \dots, r_s \rangle * \langle b_1, \dots, b_\ell \mid s_1, \dots, s_t \rangle$$

$$\langle a_1, \dots, a_k, b_1, \dots, b_\ell \mid r_1, \dots, r_s, s_1, \dots, s_t \rangle$$

Example

$$\pi_1\big((S^1\times S^1)\vee(S^1\times S^1), x\big) = \langle a,b,c,d \mid ab=ba,\ cd=dc\rangle$$

$$(\mathbb{Z}^2 * \mathbb{Z}^2)$$

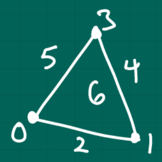
$\pi_1 = \mathbb{Z}^2$ $\qquad$ $\pi_1 = \mathbb{Z}^2$

$\langle a,b \mid ab=ba\rangle$ $\qquad$ $\langle c,d \mid cd=dc\rangle$

<u>note</u> <u>not</u> abelian

$$ac \neq ca$$

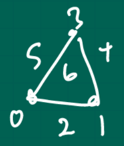
recall Let X be a Δ-complex and order the simplices.

$X =$ 

$X_0 \subset X_1 \subset X_2 \subset X_3 \subset X_4 \subset X_5 \subset X_6$

$X_i \hookrightarrow X_j$

gives $i_*: H_k(X_i) \hookrightarrow H_k(X_j)$



	X_0	X_1	X_2	X_3	X_4	X_5	X_6

H_0	X_0	X_1	X_2	X_3	X_4	X_5	X_6
X_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
X_1		$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
X_2			$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
X_3				$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
					$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
						$\mathbb{Z}$ $\mathbb{Z}$	$\mathbb{Z}$ 0
							$\mathbb{Z}$

H_1

$X_0 \hookrightarrow X_1$

$H_0(X_0) \to H_0(X_1)$

$i_* \; \mathbb{Z} \longrightarrow \mathbb{Z}^2$

$i_*(\mathbb{Z}) = \mathbb{Z}$

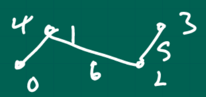
$\uparrow$

$H_0(X_0)$

table of $i_*(H_0(X_i)) \subseteq H_0(X_j)$

for $i \leq j$

Example



$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow X_4 \hookrightarrow X_5 \hookrightarrow X_6$$

H_0