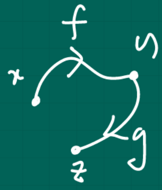
path connected / path components of X

path composition:

$f : I \longrightarrow X$

$g : I \longrightarrow X$



$f \cdot g : I \longrightarrow X$

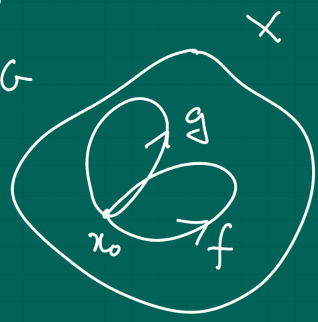
$$f \cdot g(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

<u>Recall</u> G group

operation $G \times G \longrightarrow G$

- identity
- inverses
- associativity

not necessarily abelian



<u>Fundamental group</u>

intuition

Def^{n} two loops are equivalent if they are homotopic rel endpoints

REL(ATIVE TO)

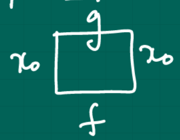
Prop^{n} This is an equivalence relation on loops

Proof · reflexive $f \sim f$

$$F(x,t) = f(x)$$

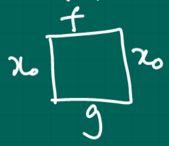
· symmetry $f \sim g \Rightarrow g \sim f$

$f \sim g \Rightarrow F : I \times I \longrightarrow X$



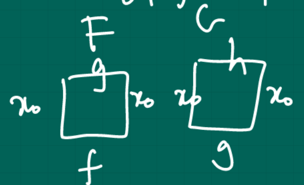
Consider $\overline{F} : I \times I \longrightarrow X$

$$\overline{F}(x,t) = F(x, 1-t)$$

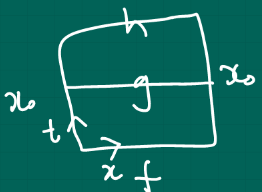


$\Rightarrow g \simeq f$

• transitive $f \simeq g$, $g \simeq h$, $f \simeq h$



$H : I \times I \longrightarrow X$



$H(x,t) =$
$F(x, 2t)$ $0 \le t \le \frac{1}{2}$
$G(x, 2t-1)$ $\frac{1}{2} \le t \le 1$

<u>Notation</u> fundamental group $\pi_1(X, x_0)$

as a <u>set</u>: equivalence classes of loops up to homotopy rel endpoints

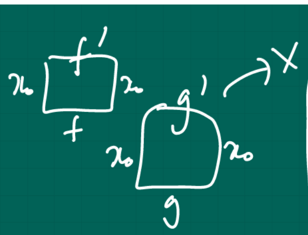
<u>operation</u>: path composition

$$[f][g] = [f \cdot g]$$

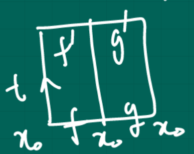
check well defined, i.e.
$f \simeq f'$, $g \simeq g'$ then $f \cdot g \simeq f' \cdot g'$

$f \simeq f'$ by F

$g \simeq g'$ by G



define $H: I \times I \to X$



$$H(x,t) = \begin{cases} F(2x,t) & 0 \le x \le \frac{1}{2} \\ G(2x-1,t) & \frac{1}{2} \le x \le 1 \end{cases}$$

So $[f][g] = [f \cdot g]$ well defined $\square$

set: loops, homotopy rel endpoints

operation: $[f][g] = [f \cdot g]$

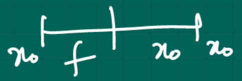
check group properties

<u>Defn</u> $e: I \to X$ constant loop
$x \mapsto x_0$

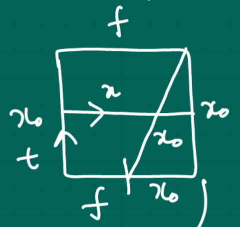
<u>Claim</u> $[e]$ is the group identity

$$[e][f] = [e] = [f][e]$$

check $\overset{[f\cdot e]}{[f][e]} = [f]$

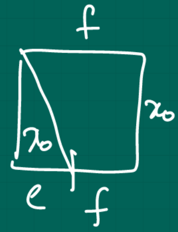


want $f\cdot e \simeq f$



$$F(x,t) = \begin{cases} x_0 & \text{if } x \geq \frac{1}{2}t + \frac{1}{2} \\ f(x(2-t)) & x \leq \frac{1}{2}t + \frac{1}{2} \end{cases}$$

also $e\cdot f \simeq f$



identity ✓

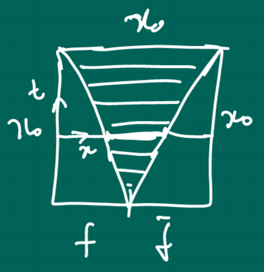
<u>inverses</u>

$f : I \longrightarrow X$

$\bar{f} : I \longrightarrow X$

$\bar{f}(x) = f(1-x)$

<u>Claim</u> $[f][\bar{f}] = [e]$



want

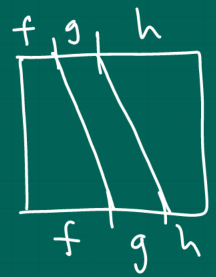
$$f \cdot \bar{f} \simeq e$$

$$F(x,t) = \begin{cases} f(2x) & 0 \le x \le \frac{1-t}{2} \\ f\left(\frac{1-t}{2}\right) & \text{otherwise} \\ \bar{f}(2x) & \frac{1+t}{2} \le x \le 1 \end{cases}$$

so $f \cdot \bar{f} \simeq e$

associativity $[f]([g][h]) = ([f][g])[h]$

$$f \cdot (g \cdot h) \simeq (f \cdot g) \cdot h$$



$\square$

Conclusion we have defined the fundamental group

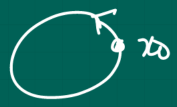
$\pi_1(X, x_0)$

Examples

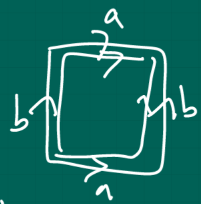
X contractible $\pi_1(X, x_0) = 1$

$\pi_1(S^2) = 1$

$\pi_1(S^1) \cong \mathbb{Z}$



$\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$



$a \cdot b = b \cdot a$

$\pi_1(S^1 \vee S^1) = F_2 = \langle a, b \mid \; \rangle$

↑ free (non-abelian) group on two generators

$F_2 = \langle a, b \mid \rangle$

set: all finite strings/lists of letters in $\{a, b, a^{-1}, b^{-1}\}$

$abbbaba^{-1}a^{-1}bbb^{-1}b^{-1}$

up to equivalence:

$u a a^{-1} v \sim uv$

$u a^{-1} a v \sim uv$

$u b b^{-1} v \sim uv$

$u b^{-1} b v \sim uv$

operation: concatenation

example

$\underbrace{aba^{-1}b^{-1}}_{g} \quad \underbrace{bbbbaa}_{h}$

$gh = aba^{-1}b^{-1}bbbbaa$

$\sim aba^{-1}bbbaa$

identity $e = \emptyset$

inverses $a a^{-1} = \emptyset = 1$

in general

$$(ababa^{-1})^{-1} = ab^{-1}a^{-1}b^{-1}a^{-1}$$

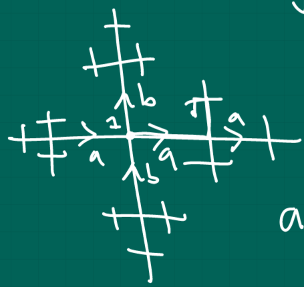
$$ababa^{-1}ab^{-1}a^{-1}b^{-1}a^{-1} = \emptyset = 1$$

associativity $\checkmark$.

$ab \neq ba$

Geometric model

infinite 4-valent tree



$aba^{-1}b^{-1}$

Thm $f: X \longrightarrow Y$ cts

s.t. $f(x_0) = y_0$

induces a homomorphism

$f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

Furthermore, $f \simeq g$

$\Rightarrow f_* = g_*$

$\gamma: I \longrightarrow X$

$[\gamma] \longmapsto [f \circ \gamma]$

$I \xrightarrow{\gamma} X \xrightarrow{f} Y$

$f \circ \gamma: I \longrightarrow Y.$