last time

homotopy $F: X \times I \to Y$

notation $f_t(x) = F(x,t)$

Thm $f, g: X \to Y$

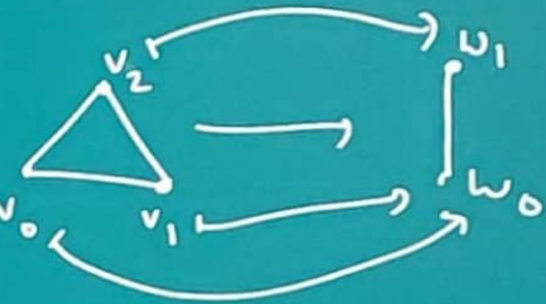
$f \simeq g \Rightarrow f_* = g_*: H_k(X) \to H_k(Y)$

$\uparrow$ homotopic

Ingredients $f: X \to Y$ is homotopic to a simplicial map $\leftarrow$ takes simplices to simplices by induced linear map

$\leftarrow$ up to subdivision of X

e.g.



Example $f: S^1 \to S^1$

$\theta \mapsto 2\theta$

$$X \qquad\qquad Y$$

$$f : S^1 \longrightarrow S^1$$

$a \longmapsto c$

$b \longmapsto c$

$v_0 \longmapsto w$

v_1

$$\boxed{\begin{aligned} \partial a &= v_0 - v_1 \\ \partial b &= v_1 - v_0 \\ \partial(a+b) &= 0 \end{aligned}}$$

$n=1$ $\qquad d=0$

$$\begin{tikzcd}
X. & 0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "\partial"] \arrow[d, "f_\#", "{[1\ 1]\ a,b}"'] & \mathbb{Z}^2 \arrow[r] \arrow[d, "f_\#", "{[1\ 1]\ v_0,v_1}"'] & 0 \\
Y & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$c \qquad\qquad w$

<u>recall</u> $f_\# : C_k(X) \to C_k(Y)$ is a chain map if diagram commutes

check exercise

chain map gives homomorphism on $H_k(X) \to H_k(Y)$

$f_* : H_0(X) \longrightarrow H_0(Y)$

$[v_0] \longmapsto [f(v_0)] = [w]$

$\mathbb{Z} \xrightarrow{\times 1} \mathbb{Z}$

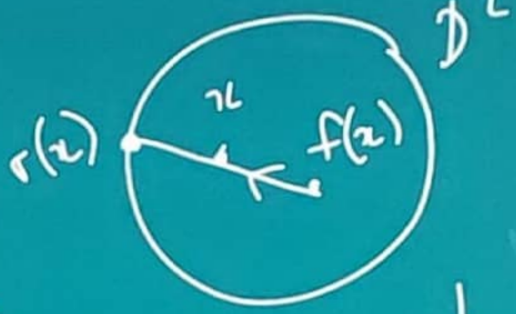
$f_* : H_1(X) \longrightarrow H_1(Y)$ $\quad [2c]$

$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ $\quad //$

$[a+b] \longmapsto [f(a)+f(b)]$

Scanné avec CamScanner

Thm [Brouwer] Every cts $f\ D^2 \to D^2$ has a fixed point



$r(x) = x$ if $x \in \partial D^2 = S^1$

Proof suppose $f : D^2 \to D^2$ has no fixed points, then it defines a <u>retraction</u> $r\ D^2 \to S^1 = \partial D^2$

ie a cts map st $r|_{\partial D^2 = S^1} = \mathbb{1}_{S^1}$

$$\begin{tikzcd} \partial D^2 \arrow[r, hook, "i"] \arrow[rr, bend right, "r\,i\, =\, \mathbb{1}_{\partial D^2}"'] & D^2 \arrow[r, "r"] & \partial D^2 \end{tikzcd}$$

$$S^1 \xrightarrow{i} D^2 \xrightarrow{r} S^1$$

$$\begin{tikzcd} H_1(S^1) \arrow[r, "i_*"] \arrow[rr, bend right, "\mathbb{1}_{H_1(S^1)}"'] & H_1(D^2) \arrow[r, "r_*"] & H_1(S^1) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[rr, bend right, "\times\ \mathbb{1}_{\mathbb{Z}}"'] & 0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$

$\square$

__Def$^{\underline{n}}$__ X, Y are __homotopy equivalent__ if there are cts maps

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

such that

$$g \circ f \simeq \mathbb{1}_X$$

$$f \circ g \simeq \mathbb{1}_Y$$

explicitly, means

$$A \quad X \times I \longrightarrow X$$

$$A(x, 0) = g(f(x))$$

and

$$A(x, 1) = x$$

__Fact__ $X \simeq Y \Rightarrow H_k(X) = H_k(Y)$ for all k

__Pf__

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "\simeq \mathbb{1}_X"] & Y \arrow[r, "g"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

$$\begin{tikzcd} H_k(X) \arrow[r, "f_*"] \arrow[rr, bend right, "\mathbb{1}_{H_k(X)}"'] & H_k(Y) \arrow[r, "g_*"] \arrow[rr, bend right, "\mathbb{1}_{H_k(Y)}"'] & H_k(X) \arrow[r, "f_*"] & H_k(Y) \end{tikzcd}$$

$$H_k(X) \xrightarrow{f_*} H_k(Y) \xrightarrow{g_*} H_k(X) \xrightarrow{f_*} H_k(Y)$$

iso iso

$\Rightarrow$ f_* injective, g_* surjective

$\Rightarrow$ g_* inj, f_* surj

$\Rightarrow$ f_* iso, g_* iso $\square$

Defn We say X is contractible if $X \simeq \{pt\}$

Remark $H_k(X) = \begin{cases} \mathbb{Z} & \text{if } k=0 \\ 0 & \text{if } k \geq 1 \end{cases}$

Examples

exercise $I^n, \mathbb{R}^n$

$[0,1] \subseteq \mathbb{R}$ is contractible

$$[0,1] \underset{g}{\overset{f}{\rightleftarrows}} \{0\} \qquad f \cdot g = \mathbb{1}_0$$

$\mathbb{1}_I \simeq g \cdot f : I \longrightarrow I$

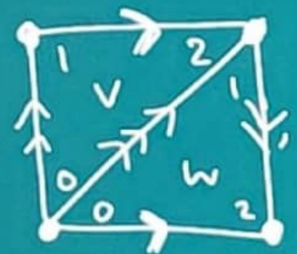
$[0,1] \longrightarrow 0$

$F : I \times I \longrightarrow I$

$(x,t) \longmapsto xt$

$f_0(x) = 0 \quad f_1(x) = x$

Klein bottle



$$\Delta = \{ [v_0, v_1, v_2], [w_0, w_1, w_2] \}$$

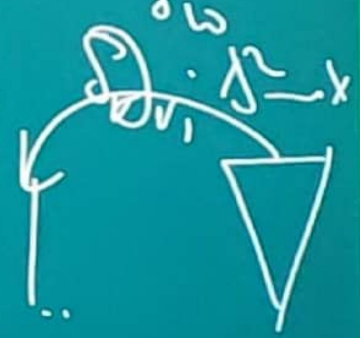
$$F = \{ [v_1, v_2] \sim [w_0, w_2], \quad [v_0, v_1] \sim [w_1, w_2], \quad [v_0, v_2] \sim [w_0, w_1] \}$$

Faces = 2

Edges = 3

Vertices ?= 1

$$[v_0] \sim [w_1] \sim [v_2] \sim [w_2] \sim [v_1] \sim \dots$$



wedge product

$X \vee Y$

$\uparrow$ pick points $x \in X$
$y \in Y$

take $X \sqcup Y / x \sim y$

<u>Example</u> $S^1 \vee S^1 =$