last time

$X = \{x_i\}_{i=1}^{n} \subseteq \mathbb{R}^d$

X_r = Rips complex

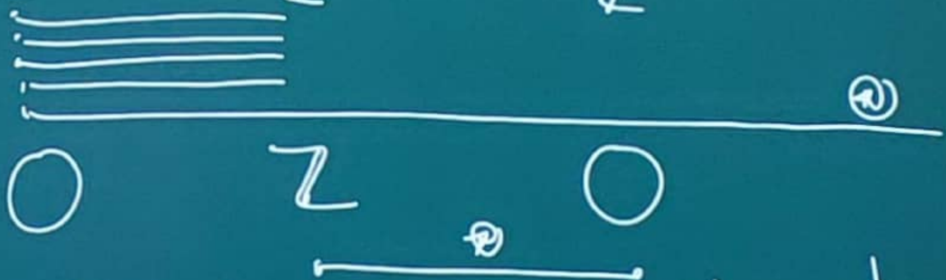
- edges (x_i, x_j) if $d(x_i, x_j) \leq r$
- fill in triangles, tetrahedra, etc gives $X_{r_1} \subset X_{r_2} \subset \cdots$

Example

$X_0 \hookrightarrow X_1 \hookrightarrow X_2$

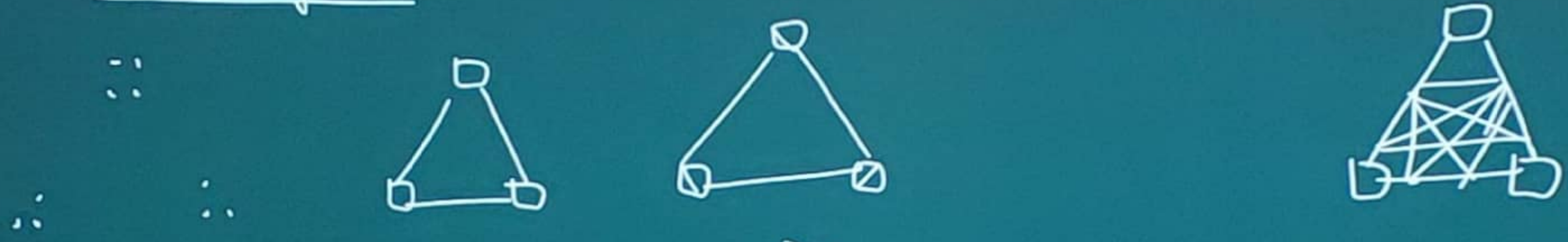
	$r=0$	$r=1$	$r=2$

H_0	$\mathbb{Z}^5$	$\mathbb{Z}$	$\mathbb{Z}$
H_1	0	$\mathbb{Z}$	0



interesting homology $\longleftrightarrow$ long intervals

Example



	X_0	X_1	X_2	X_{10}
H_1	0	$\mathbb{Z}^4$	$\mathbb{Z}$	0

c_k

Technical notes

- R probably uses $\mathbb{Z}_2$ coefficients not $\mathbb{Z}$

$\sum n_i \sigma_i \quad n_i \in \mathbb{Z}_2$ $\qquad$ $\sum n_i \sigma_i \quad n_i \in \mathbb{Z}$

- $X = \{x_i\}_{i=1}^{n} \subseteq \mathbb{R}^d$

$X_r \simeq \Delta^{n-1}$ for r large enough

Δ^{n-1} contains $2^{n-1} - 1$ subsimplices

— can't compute everything

n vertices

$\binom{n}{2}$ edges $\sim n^2$

$\binom{n}{3}$ faces $\sim n^3$

can compute H_0, H_1, H_2

Questions | problems

- errors? stability Thm ✓
- noise? — clean up data e.g. (choose points with many nearest neighbours)

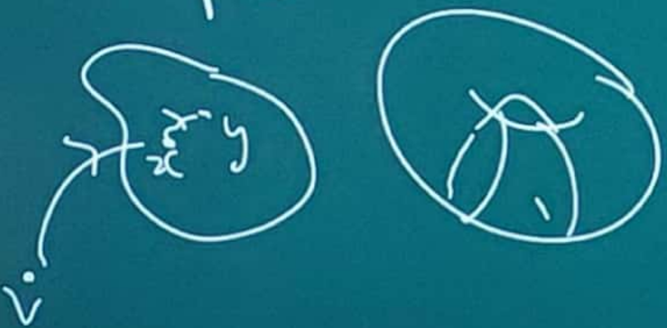
- significance? — subsample probability measure

Homotopy

idea $f : X \to Y$ cts

want continuously varying families of maps

examples



Defn A homotopy is a map

$$F : X \times I \longrightarrow Y$$

cts

notation $f_t(x) = F(x,t)$

A homotopy between $f : X \to Y$
$g : X \to Y$

is a map $F : X \times I \longrightarrow Y$

s.t. $\cdot\, f_0(x) = f(x)$

$f_1(x) = g(x)$

$I = [0,1]$

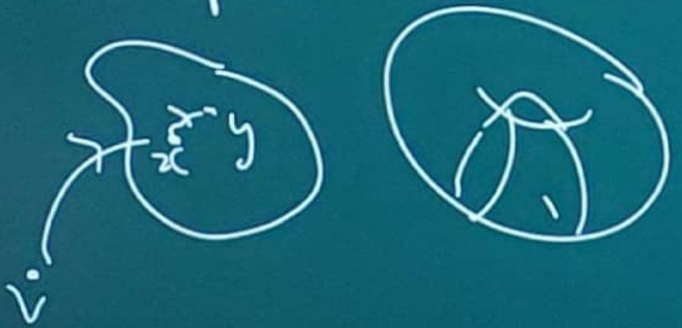
$0 \quad 1 \quad \subset \mathbb{R}$

Homotopy

idea $f : X \to Y$
cts

want continuously varying families of maps

examples



Def$^{\underline{n}}$ A homotopy is a map

$$F : X \times I \longrightarrow Y$$

cts

notation $f_t(x) = F(x,t)$

A homotopy between $f : X \to Y$
$g : X \to Y$
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s.t. $f_0(x) = f(x)$
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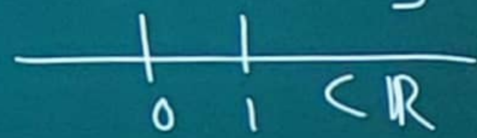
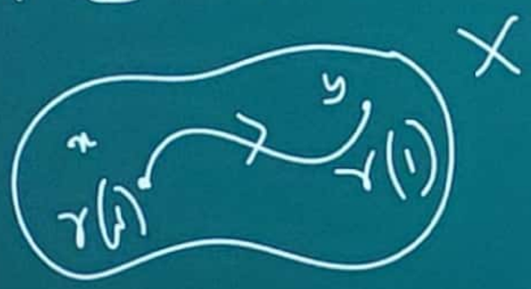


This is interesting
even for $X = \{pt\}$

$F \quad \underbrace{\{pt\} \times I}_{\cong\ I} \longrightarrow Y$

also called <u>paths</u>

$\gamma \; I \longrightarrow Y$



$x \sim y$

<u>Define</u> an equivalence relation on X

$x \sim y$ if there is path $\gamma \; I \longrightarrow X$
s.t. $\gamma(0) = x \quad \gamma(1) = y$

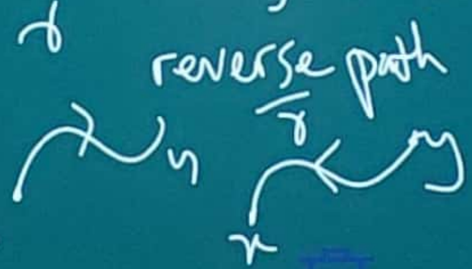
<u>check</u> • reflexive $x \sim x$

$\gamma \cdot \underset{[0,1]}{I} \longrightarrow x$ constant path
$\Rightarrow x \sim x$ ✓

• symmetry $x \sim y \Rightarrow y \sim x$

$x \sim y \Rightarrow \exists\ \gamma \cdot I \longrightarrow X$ with $\gamma(0) = x$
$\gamma(1) = y$

reverse path $\bar{\gamma} \cdot I \longrightarrow X \quad \bar{\gamma}(t) = \gamma(1-t)$



$\Rightarrow y \sim x$ ✓

· transitive $x\sim y,\ y\sim z \Rightarrow x\sim z$

then $\gamma_1\cdot\gamma_2$ is a path from x to z so

$x\sim z$ ✓

notation $f\circ g$

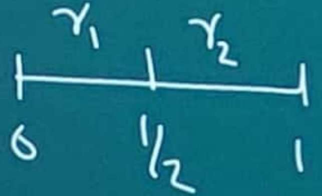
function composition

define $\gamma_1\cdot\gamma_2$

path composition

$$I \longrightarrow X$$

$$t \longmapsto \begin{cases} \gamma_1(2t) & 0\le t\le \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2}\le t\le 1 \end{cases}$$



note this is cts as

$$\gamma_1(1)=\gamma_2(0)$$

so this is an equivalence relation □

Def^n equivalence classes are called the path components of X

Examples

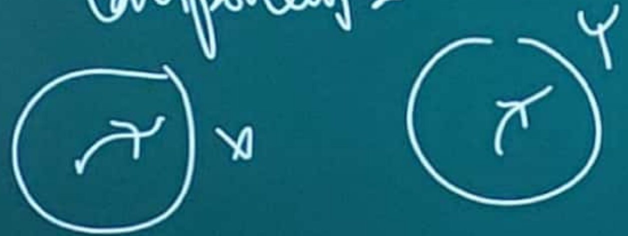
-  S^1  S^n $S^1 \times S^1$

↑ 1-path component

$\Longleftrightarrow$ X is path connected

Observation $X \overset{\uparrow \text{ homeo}}{\cong} Y$

$\Rightarrow$ same # of path components



- $S^0 = \cdot \quad \cdot$ $\{x, y\}$

Prop^n: Two path components

Proof spose $\gamma : I \longrightarrow S^0$ s.t. $\gamma(0) = x$ and $\gamma(1) = y$, γ cts so f^{-1}(open set) is open. $\{x\}$, $\{y\}$ open in S^0.

Consider $\gamma^{-1}(\{x\}), \gamma^{-1}(\{y\})$

$[0,1]$ $\quad 0 \longmapsto 1$

$0 \in U = \gamma^{-1}(\{x\})$ $\qquad$ $1 \in V = \gamma^{-1}(\{y\})$

$[0,1] =$ 

$U \cap V = \emptyset$

$\sup(U) =$ least upper bound $= s$

for all $t \in U$ $t \leq s$

and no other upper bound $< s$

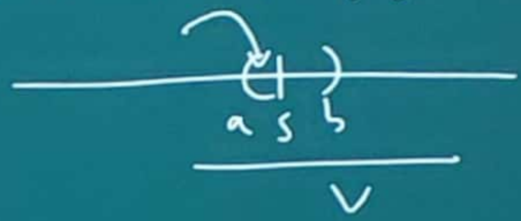
Case 1) $s \in U$ 

there is $s \in (a,b) \subseteq U$

so $\exists\, t \in U$ $t > s$ ~~#~~ upper bound

Case 2 $s \in V$

there is $s \in (a,b) \subseteq V$



so $\exists\, t < s$ which is also an upper bound for U ✗

least upper bound

so no cts map $I \longrightarrow \{x,y\}$

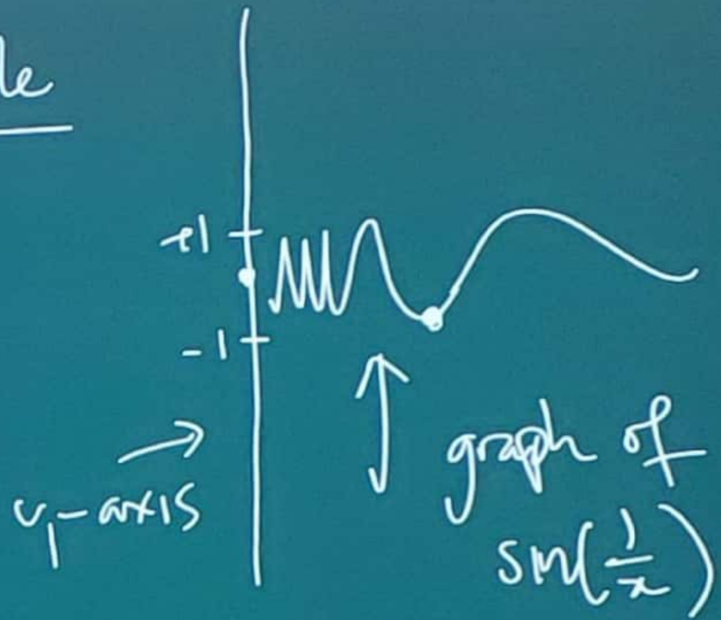
Fact discrete set of n points has n path components

Warning connected $\neq$ path connected

Defn
X is connected if it is not the disjoint union of two open sets

Last $\Rightarrow [0,1]$ connected

Example

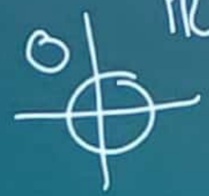


$X =$ union of y-axis and graph of $\sin(\frac{1}{x}) \subseteq \mathbb{R}^2$ with induced topology

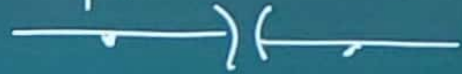
X not Δ-complex

Scanné avec CamScanner

Application



$\mathbb{R}\setminus\{0\}$ $(-\infty, 0)$ $(0, \infty)$



not path connected

Thm $\mathbb{R}^1 \not\cong \mathbb{R}^2$

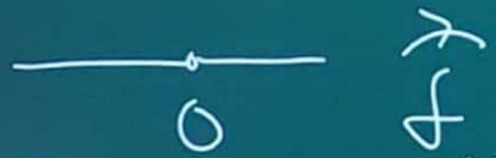
$\mathbb{R}^2\setminus\{0\}$ is path connected

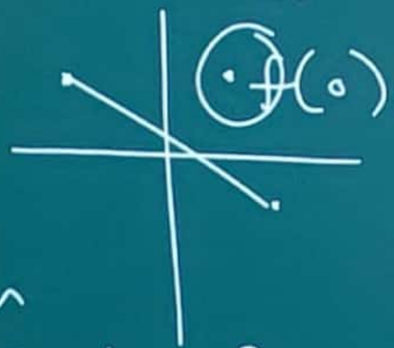
Proof suppose $f: \mathbb{R}^1 \to \mathbb{R}^2$ is a homeomorphism

so $\mathbb{R}^1 \not\cong \mathbb{R}^2$ □

$\mathbb{R}^2$ H_1 $\mathbb{Z}$

$\mathbb{R}$ $\mathbb{Z}^2$ th. $\mathbb{Z}$





$\mathbb{R}^3\setminus\{pt\}$ H_1 0

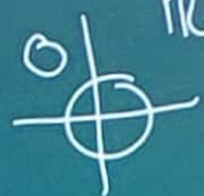
gives a homeomorphism

f $\mathbb{R}\setminus\{0\}$ to $\mathbb{R}^2\setminus\{f(0)\}$

Thm $\mathbb{R}^a \cong \mathbb{R}^b \Rightarrow a = b$

Proof use homology □

Application

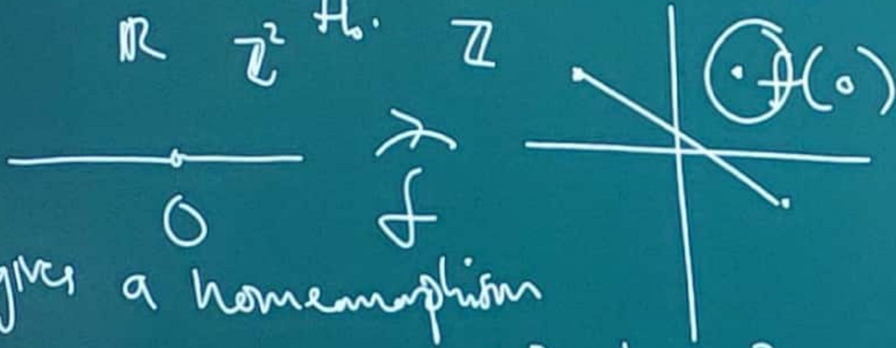


Thm $\mathbb{R}^1 \not\cong \mathbb{R}^2$

Proof suppose $f\ \mathbb{R}^1 \to \mathbb{R}^2$ is a homeomorphism

$\mathbb{R}$ $\mathbb{Z}^2$ H_1 $\mathbb{Z}$

$\mathbb{R}^2$ H_1 $\mathbb{Z}$



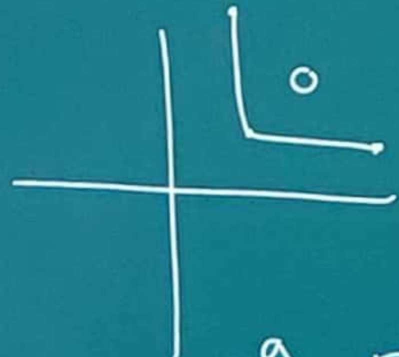
gives a homeomorphism

f $\mathbb{R}\setminus\{0\}$ to $\mathbb{R}^2\setminus\{f(0)\}$

$\mathbb{R}\setminus\{0\}$ $(-\infty,0)$ $(0,\infty)$

not path connected

$\mathbb{R}^2\setminus\{0\}$ is path connected



so $\mathbb{R}^1 \not\cong \mathbb{R}^2$ $\square$

$\mathbb{R}^3\setminus\{p\}$

H_1 0

Thm $\mathbb{R}^a \cong \mathbb{R}^b \Rightarrow a=b$

Proof use homology $\square$