want $f: X \longrightarrow Y$ cts

gives $f_*: H_k(X) \longrightarrow H_k(Y)$

homomorphism

<u>example</u> $\iota: X \hookrightarrow Y$

X Δ-subcomplex of Y

induces $i_*: H_k(X) \to H_k(Y)$

<u>Defn 2</u> A chain map is a collection of ~~maps~~ homomorphisms $f_k: C_k(X) \to C_k(Y)$

$$\begin{array}{ccccccc} \xrightarrow{\partial} & C_k(X) & \xrightarrow{\partial_k^X} & C_{k-1}(X) & \to & - \\ & \downarrow f_k & & \downarrow f_{k-1} & & \\ \xrightarrow{\partial} & C_k(Y) & \xrightarrow{\partial_k^Y} & C_{k-1}(Y) & \to & \end{array}$$

such that this diagram commutes, ie

$$\partial_k^Y f_k = f_{k-1} \partial_k^X$$

<u>Propn</u> A chain map induces

$$f_*: H_k(X) \longrightarrow H_k(Y)$$

Proof

$z + \mathrm{im}(C_{k+1})$

$$H_k(X) \ni [z] \qquad z \text{ cycle in } C_k(X)$$

$$\downarrow \qquad \downarrow \qquad \text{i.e. } \partial z = 0$$

$$\text{want } \partial f(z) = 0 \checkmark$$

$$H_k(Y) \qquad [f_k(z)] \qquad \qquad \begin{array}{c} \| \\ f\underbrace{\partial z}_{=0} = 0 \end{array}$$

$$\begin{tikzcd}
\overset{y}{C_{k+1}(X)} \arrow[r, "\partial"] \arrow[d, "f_{k+1}"] & \overset{z}{C_k(X)} \arrow[r, "\partial"] \arrow[d, "f_k"] & \overset{\partial z}{C_{k-1}(X)} \arrow[d, "f_{k-1}"] \\
C_{k+1}(Y) \arrow[r, "\partial"] & C_k(Y) \arrow[r, "\partial"] & C_{k-1}(Y)
\end{tikzcd}$$

$$fy \longmapsto \partial f y \qquad fz \qquad \partial f z \quad f\partial z$$

check well defined ⊛

$$[fz] \checkmark$$

$$\|$$

Def$^{\underline{n}}$ $\quad f_*[z] = [f(z)]$

⊛ Consider $f_*[z + \partial y] = [f(z + \partial y)] = [f(z) + f\partial y] = [fz + \partial f y]$

$\square$

check inclusion

$X \hookrightarrow Y$

gives chain map

$$\begin{tikzcd} C_k(X) \arrow[r, "\partial"'] \arrow[d, hook, "\iota"] & C_{k-1}(X) \arrow[d, hook, "\iota"] \\ C_k(Y) \arrow[r, "\partial"'] & C_{k-1}(Y) \end{tikzcd}$$

✓

Persistent Homology

data set $X = \{x_i\}_{i=1}^{n} \subseteq \mathbb{R}^d$

$B(x_i, r)$

ball of radius r

idea define

$$X_r = \bigcup_{i=1}^{n} B(x_i, r) \rightsquigarrow H_k(X_r)$$

problem hard to compute

$X_r = \bigcup B(x_i, r)$

solution use Rips complex

X_r = · take edges $[x_i, x_j]$ if $d(x_i, x_j) \le r$

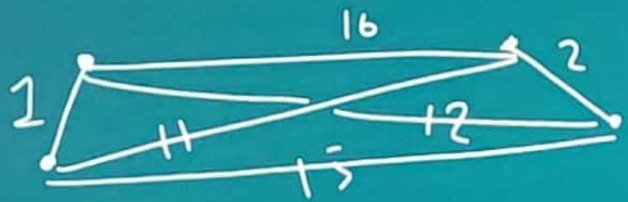
· fill in all possible simplices,

i.e. add $[x_{i_1}, \dots, x_{i_k}]$ if distances all $\le r$

Example

Fact works ok in practice

Example



$X_0 \subset X_1 \subset X_2 \subset \quad X_{10} \subset X_{11} \subset X_{12} \subset X_{13}$

consider H_0

$\mathbb{Z}^4 \to \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2 \longrightarrow \quad \mathbb{Z} \longrightarrow \mathbb{Z} \to \mathbb{Z} \longrightarrow \mathbb{Z}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \qquad \text{need det} = \pm 1$$

$r_2 = r_2 + 2r_1$ ✓

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$r_2 = 2r_2 + r_1$ X

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$

↑ not in $\mathbb{Z}$