plan:

- homology
- persistent homology
- homotopy

recall

X Δ-complex

$\rightsquigarrow H_k(X)$ homology groups

Examples

$X = \{\text{point}\}$

$[v]$

$$0 \to 0 \to \mathbb{Z} \xrightarrow{d=0} 0$$

$$H_k(\{pt\}) = \begin{cases} \mathbb{Z} & \text{if } k=0 \\ 0 & \text{if } k \geq 1 \end{cases}$$

Prop$^{\underline{n}}$ $H_0(X) \cong \mathbb{Z}^{\#\text{components}}$

Pf $X =$ 

path $\gamma = e_1 + e_2 + e_3$

connects v to w

$$\begin{array}{ccccccc} C_2 & \longrightarrow & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & 0 \\ & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ & & \gamma & & v, w & & \end{array}$$

$\partial \gamma = w - v$

recall

$\partial[v_0, v_1] = [v_1] - [v_0]$

$H_0(X) = \ker(\partial_0) \big/ \operatorname{Im}(\partial_1)$

$= C_0 / \operatorname{im}(\partial_1)$

$[v] = v + \operatorname{Im}(\partial_1)$

$[w] - [v] = 0$ in $H_0(X)$ i.e.

$H_0(X) = \mathbb{Z}^a$

$a = \#$ components

generated by a choice of vertex in each component

$[v] = [w]$

<u>Example</u>

$X = S^1$

$0 \to C_1 \xrightarrow{\partial_1} C_0 \to 0$

$0 \to \underset{e}{\mathbb{Z}} \xrightarrow{0} \underset{v}{\mathbb{Z}} \to 0$

$\Delta = \{[v_0, v_1]\}$

$F = \{[v_0] \sim [v_1]\}$

$\partial e = v - v = 0$

$\partial[v_0, v_1] = [v_1] - [v_0]$

$H_0(S^1) = \mathbb{Z}$ $\quad [v]$

$H_1(S^1) = \mathbb{Z}$ $\quad [e]$

$H_k(S^1) = 0 \quad k \geq 2$

useful fact

$[z] \in H_1(X)$ can be realised by loops

Example

X^0 

$X = S^2$ $\quad X^1$ 

$X^2 = S^2$

$$0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

$$\mathbb{Z}^2 \qquad \mathbb{Z}^3 \qquad \mathbb{Z}^3$$

$$v, w \qquad a, b, c \qquad x, y, z$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Delta = \{ \overset{v}{[v_0, v_1, v_2]}, \overset{w}{[w_0, w_1, w_2]} \}$$

$$\partial v = b - c + a$$

$$\partial [v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

$$F = \left\{ \begin{array}{ll} [v_0, v_1] \sim [w_0, w_1], & a \\ [v_1, v_2] \sim [w_1, w_2], & b \\ [v_0, v_2] \sim [w_0, w_2] & c \end{array} \right\}$$

$$\partial w = b - c + a$$

$$\partial a = y - x$$

$$\partial [v_0, v_1] = [v_1] - [v_0]$$

$$0 \to \mathbb{Z}^2 \xrightarrow{D_2} \mathbb{Z}^3 \xrightarrow{D_1} \mathbb{Z}^3 \to 0$$

$u, v \qquad a, b, c \qquad x, y, z$

$$D_1 D_2 = 0$$

$$L_1 D_1 U_1 U_1^{-1} L_2^{-1} L_2 D_2 U_2 = 0$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$r_3 = r_3 + r_1 + r_2$

$L_2 D_2 U_2 \qquad L_1 D_1 U_1$

$$H_0(S^2) \cong \mathbb{Z}$$

$$H_1(S^2) = \mathbb{Z}/\mathbb{Z} = 0$$

$$H_2(S^2) = \mathbb{Z} \leftarrow [u - v]$$

$$H_k(S^2) = 0$$

Constructions

$X \vee Y$ wedge product

$(X, x_0) \sqcup (Y, y_0) \big/ x_0 \sim y_0$

Ex

$S^1 \vee S^1$

$S^2 \vee S^1$

if X, Y Δ-complexes, x_0, y_0 vertices

$\rightsquigarrow$ $X \vee Y$ is a Δ-complex

$S^2 \vee S^1$

CX cone on X

CX

x

$= X \times I \big/ (x,1) \sim (y,1)$ for all $x, y \in X$

note cone on simplex is simplex

$[v_0, v_1, v_2]$

$\downarrow$

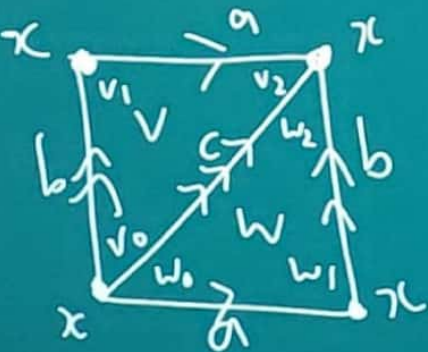
$[x, v_0, v_1, v_2]$

$SX =$  X

$= X\times I \Big/ \begin{matrix}(x,0)\sim(y,0) \ \forall x,y \\ (x,1)\sim(y,1)\end{matrix}$

$X \times Y$

Example $T^2 = S^1 \times S^1$



$$0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow[0]{\partial_1} C_0 \to 0$$

$$\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix} \sim \quad \mathbb{Z}^2 \ \begin{array}{c|c} & v,\ w \\ \hline \begin{matrix}a\\b\\c\end{matrix} & \begin{bmatrix}1&1\\1&1\\-1&-1\end{bmatrix}\end{array} \ \mathbb{Z}^3 \qquad \mathbb{Z} \ \begin{bmatrix}0&0&0\end{bmatrix} x$$

$$\partial[v_0,v_1] = [v_1] - [v_0]$$
$$x - x = 0$$

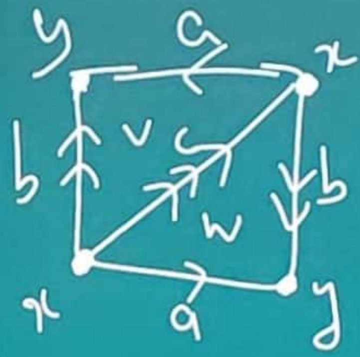
$$\partial[v_0,v_1,v_2] = [v_1,v_2] - [v_0,v_2] + [v_0,v_1]$$
$$\partial v = a - c + b$$
$$\partial w = b - c + a$$

$$H_0(T^2) = \mathbb{Z}, \quad H_2(T^2) = \mathbb{Z}$$
$$H_1(T^2) = \mathbb{Z}^2$$

Example $\mathbb{RP}^2$

S^2  $/$ antipodal map $\sim$ 

$\sim$ $D^2/\sim$ antipodal map on $\partial D^2 = S^1$



$$0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z}^2 \to 0$$

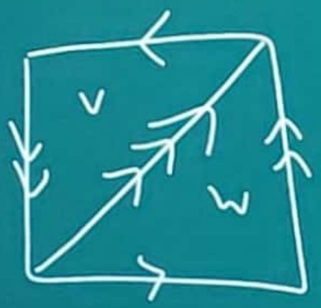
$v, w \qquad a, b, c \qquad x, y$

$$\partial v = c + a - b$$
$$\partial w = c + b - a$$
$$\partial a = y - x$$
$$\partial b = y - x$$
$$\partial c = x - x = 0$$

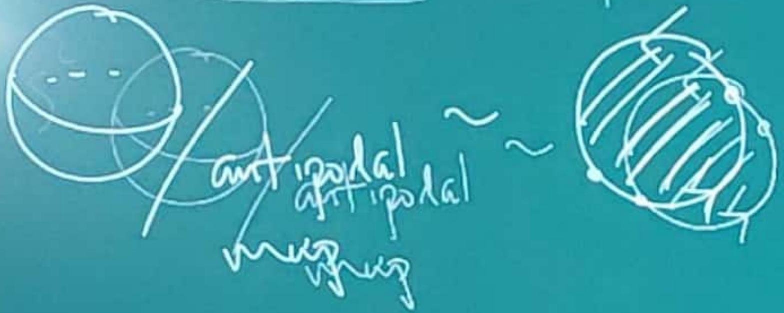
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}_2 \qquad \begin{bmatrix} 1 & \underbrace{0 \quad 0}_{} \\ 0 & 0 \quad 0 \end{bmatrix}$$

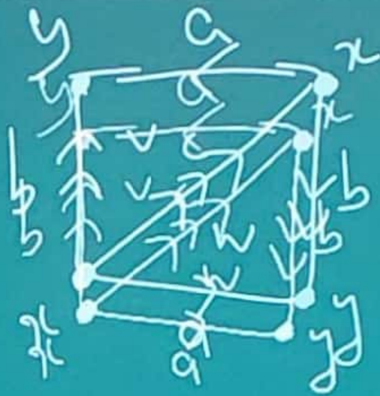
$$H_1 = \mathbb{Z} \oplus \mathbb{Z} \Big/ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \mathbb{Z}/2\mathbb{Z}$$



Example $\mathbb{RP}^2$



$\sim D^2/\sim$ antipodal map on $\partial D = S^1$



$$0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z}^2 \to 0$$

$v, w \qquad a, b, c \qquad x, y$

$\partial v = c + a - b$

$\partial w = c + b - a$

$\partial a = y - x$

$\partial b = y - x$

$\partial c = x - x = 0$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$H_1 = \mathbb{Z} \oplus \mathbb{Z} \Big/ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \mathbb{Z}/2\mathbb{Z}$$

Scanné avec CamScanner

$$0 \to \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0} 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H_0(\mathbb{RP}^2) = \mathbb{Z}^2 \Big/ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbb{Z}$$

$$H_1(\mathbb{RP}^2) = \mathbb{Z}^2 \Big/ \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} = \mathbb{Z}/2\mathbb{Z}$$

$$H_2(\mathbb{RP}^2) = 0$$

$$H_k(\mathbb{RP}^2) = 0 \quad k > 2$$

$$\underset{\substack{\| \\ S^1}}{\partial D^2} \hookrightarrow D^2$$

want

$$X \xrightarrow[\text{cts}]{f} Y \qquad H_k(X) \xrightarrow{f_*} H_k(Y)$$

homomorphism

$X \subseteq Y$ subcomplex

inclusion map $\iota: X \hookrightarrow Y$

induces a homomorphism

$$\iota_*: H_k(X) \longrightarrow H_k(Y)$$

$$H_1(S^1) \to H_1(D^2)$$

$$\mathbb{Z} \longrightarrow 0$$