Last time

- classification of surfaces

g: 0   1   2

χ: 2   0 -2

$\mathbb{R}P^2$ K

$\boxed{\chi = 2-2g}$

Fact . every surface has a Δ-complex structure

Def^n The Euler characteristic

$$\chi(X) = V - E + F$$

$V = \#$ vertices

$E = \#$ edges

$F = \#$ faces

Thm X, Y Δ-complexes $X \cong Y$ then

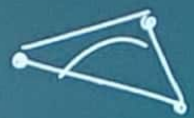
$$\chi(X) = \chi(Y)$$

More generally

$$\chi(X) = \sum_{i=0}^{n} (-1)^i c_i$$

$c_i = \#$ i-simplices

S^2



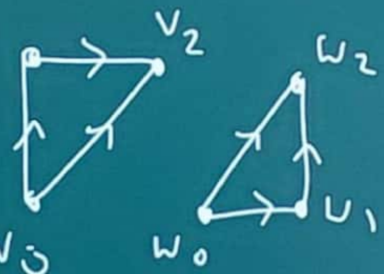
$V - E + F$

$3 - 3 + 2 \quad \chi = 2$

$4 - 6 + 4 \quad \chi = 2$

$T^2 = S^1 \times S^1$

$\Delta = \{$



$\}$

$F = \{ [w_0, w_1] \sim [v_1, v_2], [v_0, v_1] \sim [w_1, w_2], [v_0, v_2], [w_0, w_2] \}$

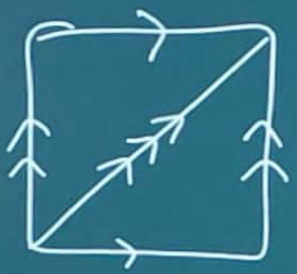
$$\chi(X) = \overset{1}{V} - \overset{3}{E} + \overset{2}{F}$$

$$[v_0] \sim [w_0] \sim [v_1] \sim [w_2] \sim [v_2] \sim [w_1]$$

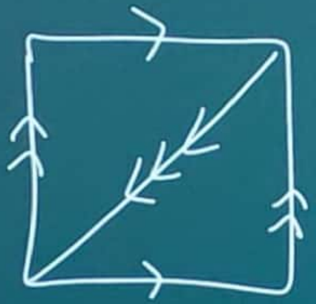
$$\chi(T^2) = 0$$

Warning

- $T^2 =$

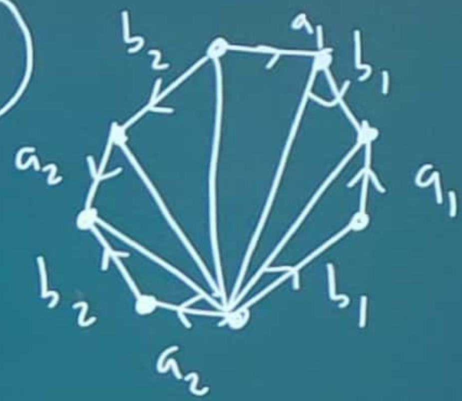


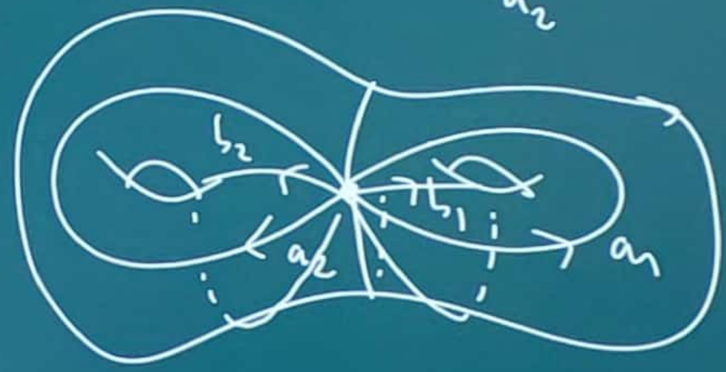
✓ works



✗ doesn't work

$g = 2$





$$\chi = V - E + F$$

$$1 - \underbrace{\frac{18}{2}}_{9} + 6 = -2$$

top spaces $X \xrightarrow[\text{cts}]{f} Y$

$\rightsquigarrow$ new invariant

$$H_k(X) \xrightarrow{f_*} H_k(Y)$$

abelian group

homomorphism

Fact

$H_1(S^1) \cong \mathbb{Z}$

$H_1(D^2) \cong 0$

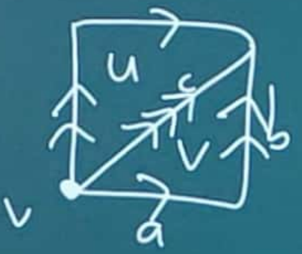
$X = \{x_i\} \subseteq \mathbb{R}^2$

$X_r = N_r(X)$

X Δ-complex

v: 1 0-simplex
e: 3 1-simplex
f: 2 2-simplex

C_i = free abelian group generated by simplices of dim i

$T^2 =$ 

$\rightsquigarrow$ sequence of abelian groups

chain groups

$$\cdots \longrightarrow C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \longrightarrow 0$$

$$\mathbb{Z}^2 \qquad \mathbb{Z}^3 \qquad \mathbb{Z}$$

$u, v \qquad a, b, c \qquad v$

$-u - v \qquad 9a - 4b + 2c \qquad 7v$

i.e we are taking formal sums of simplices

Intuition

1-dim spaces

I

$\partial I = \{a, b\}$

↑
boundary

S^1

$\partial S^1 = \emptyset$

empty set

2 dim spaces

$\mathbb{R}^2_+$

D^2

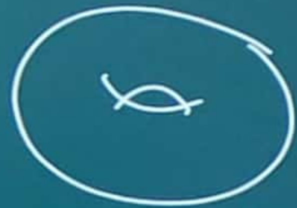
$\partial D^2 = S^1$

S^2

$\partial S^2 = \emptyset$

$\partial X = S^1 \sqcup S^1 \sqcup S^1$

$\partial \partial X = \emptyset$



$\partial T^2 = \emptyset$

3d spaces

B^3

$x^2+y^2+z^2 \leq 1$

$\partial B^3 = x^2+y^2+z^2 = 1$

$= S^2$

$\mathbb{R}^3_+$

$= \{x,y,z \mid z \geq 0\}$

solid torus

$D^2 \times S^1$

boundary $\partial(D^2 \times S^1) = S^1 \times S^1$

$\partial = \Sigma_2$

$\partial\partial M^3 = \emptyset$

$\mathbb{Z}^a \xrightarrow{\partial} \mathbb{Z}^b \xrightarrow{\partial} \mathbb{Z}^c$

G group

$|G| =$ size of G

$=$ order of G

$=$ number of elements

Example

$\mathbb{Z} \xrightarrow{f} \mathbb{Z}$

$1 \longmapsto 2$

$\mathbb{Z}/f(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$

$\mathbb{Z}\oplus\mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}$

$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$

$(1,0) \longmapsto (2,0)$

$(0,1) \longmapsto (0,3)$

problem: map not diagonal

use row/col operations to make it diagonal

image $f(\mathbb{Z}\oplus\mathbb{Z})$

$(a,b) \longmapsto (2a,3b)$

cosets $(0,0) + 2\mathbb{Z}\oplus 3\mathbb{Z}$

$(1,0)$ $(0,2)$ $(1,2)$

$(0,1)$ $(1,1)$