Last time

$$\alpha \quad \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$$

homomorphism

$$\alpha(x+y) = \alpha(x) + \alpha(y)$$

$n \in \mathbb{Z} \qquad \alpha(nx) = n\alpha(x)$

$\alpha \Leftrightarrow$ matrix A $(m \times n)$ matrix

$$m \begin{bmatrix} a_{11} & a_{12} & - \\ \cdots & & \\ & n & \end{bmatrix}$$

choose generating sets

$$\begin{array}{ccc} \alpha \quad \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^m \\ \uparrow & & \uparrow \\ e_1, \ldots, e_n & & f_1, \ldots, f_m \end{array}$$

standard generating set

$(1, 0, 0, \ldots 0)$

$(0, 1, 0, \ldots 0)$

$\vdots$

$(0, 0, 0, \ldots, 1)$

$\alpha: \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$

$e_1, \dots, e_n \quad f_1, \dots, f_m$

$\alpha(e_1) = a_{11} f_1 + a_{21} f_2 + \dots + a_{m1} f_m$

$$A = \begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & & \\ & & \end{bmatrix}$$

$$A = \begin{bmatrix} | & | & & | \\ \alpha(e_1) & \alpha(e_2) & \cdot & \alpha(e_n) \\ | & | & & | \end{bmatrix}$$

<u>Recall</u>

SAT

$\alpha: V \xrightarrow{A} W$

can change basis

$\alpha: V \xrightarrow{A} V$

only one basis

SAS^{-1}

$\alpha \quad \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$

$e_1, \dots, e_n \qquad f_1, \dots, f_m$

replace e_2 by $e_2 + e_1$

Q: How does this change A

$$\begin{bmatrix} | & | & | & & | \\ \alpha(e_1) & \alpha(e_1+e_2) & \alpha(e_3) & \cdot & \alpha(e_n) \\ | & | & | & & | \end{bmatrix}$$

$\alpha(e_1+e_2) = \alpha(e_1) + \alpha(e_2)$

this is a column operation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$$

$\underbrace{A\,U}$ upper triangular 1s on diagonal

Can make A upper triangular, pivots $\in \mathbb{Z}$

$$\begin{bmatrix} 4 & * & * \\ 0 & 5 & * \\ & & \ddots \end{bmatrix}$$

$\alpha \; \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$

$e_1, \ldots, e_n \qquad f_1, \ldots, f_m$

change basis here

$f_1, f_2 + f_1, f_3, \ldots, f_m$

new basis

$f_1, f_2', f_3, \ldots, f_m$

$f_2' = f_2 + f_1$

$$\begin{bmatrix} | & & \\ \alpha(e_1) & \alpha(e_2) & \cdots \\ | & & \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \cdot \\ & \cdot & \end{bmatrix}$$

$\alpha(e_1) = a_{11} f_1 + a_{21} f_2 +$

becomes

$\alpha(e_1) = a_{11} f_1 + a_{21}' f_2' + a_{31} f_3 + \cdots$

$\alpha(e_1) = a_{11} f_1 + (a_{21}')(f_1 + f_2) + \cdots$

$= (a_{11} + a_{21}') f_1 + a_{21}' f_2 + \cdots$

row operation

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ a-c & d-b \end{bmatrix}$$

Scanné avec CamScanner

$$\alpha \quad \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^m$$

change basis — change basis

$$\boxed{L\,A}\,U = D$$

L: lower triangular $\Updownarrow$ row ops

U: upper triangular $\Updownarrow$ col ops

can choose L, U to make $LAU =$ diagonal matrix.

$$\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c+na & d+nb \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ 0 & c & 1 & \\ n & & & \ddots \\ & & & & 1 \end{bmatrix}$$

inverse

$$\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example

$\alpha \quad \mathbb{Z}^2 \xrightarrow{\quad} \mathbb{Z}^2$

$(a,b) \quad A$

$A = \begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix}$

$ae_1 + be_2 \xrightarrow{\alpha} \alpha(ae_1 + be_2)$

$\begin{bmatrix} a \\ b \end{bmatrix} \qquad A\begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix} \leftarrow r_2 + r_1$

$\sim \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$\alpha(\mathbb{Z}^2) \subset \mathbb{Z}^2$?

$\alpha(\mathbb{Z}^2) = \{(2n, 0) \mid n \in \mathbb{Z}\}$

$\mathbb{Z}^2 / \alpha(\mathbb{Z}^2) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$

notation $\mathbb{Z}_2 \oplus \mathbb{Z}$

Example

$$\alpha : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$

$$\begin{bmatrix} 3 & 4 \\ -2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \quad c_2 - c_1$$

$$\sim \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \quad c_1 \leftrightarrow c_2$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad c_2 - 3c_1$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad c_2 \times -1$$

$\leftarrow$ working in $\mathbb{Z}$

$$\alpha(\mathbb{Z}^2) \subset \mathbb{Z}^2$$

$$\left\{ (k, 2l) \;\middle|\; \begin{array}{l} k \in \mathbb{Z} \\ l \in \mathbb{Z} \end{array} \right\}$$

$$\mathbb{Z}^2 / \alpha(\mathbb{Z}^2) \cong \mathbb{Z}_2$$

$$\alpha : \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{D} \mathbb{Z} \oplus \mathbb{Z}$$

$$\underbrace{\mathbb{Z} \oplus \mathbb{Z}}_{} \big/ \alpha(\mathbb{Z}^2)$$

$$\underbrace{\underbrace{\mathbb{Z}/1\mathbb{Z}}_{0} \oplus \mathbb{Z}/2\mathbb{Z}}_{\mathbb{Z}/2\mathbb{Z}}$$

Def$^{\underline{n}}$ A chain complex is a sequence of free abelian $\cong \mathbb{Z}^r$ groups G_n and homomorphisms

$$\cdots \to G_{n+1} \xrightarrow{\alpha_{n+1}} G_n \xrightarrow{\alpha_n} G_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \to G_0 \to 0$$

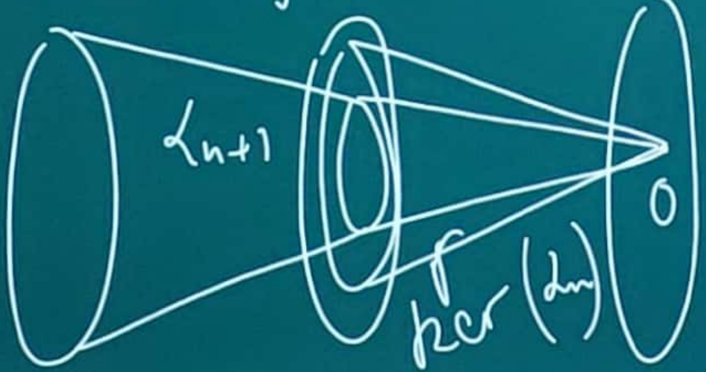
such that for all n $\alpha_n \alpha_{n+1} = 0$

$$\alpha_n \alpha_{n+1} = 0$$

$$\Rightarrow \operatorname{image}(\alpha_{n+1}) \subseteq \ker(\alpha_n)$$

$\neq$ in general

$$\boxed{H_n = \ker(\alpha_n) \big/ \operatorname{image}(\alpha_{n+1})}$$



Def$^{\underline{n}}$ n-th homology group of the chain complex

<u>Recall</u>

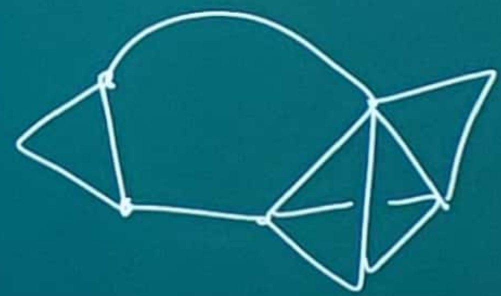
Topological spaces → algebraic invariants

simplicial complexes
(Δ-complexes from Hatcher)

<u>Intuition</u>

take some triangles,
glue edges in pairs,
get a surface

← triangles give S^2



Def^n an n-simplex

is the convex hull of $(n+1)$

points $(v_0, v_1, \ldots, v_n)$ in $\mathbb{R}^d$

$d \geq n+1$

in general position $\leftarrow$ means the

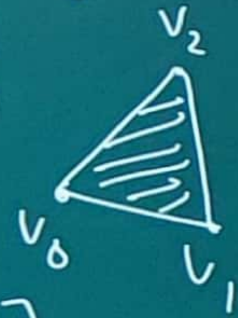
vectors $v_k - v_0 \quad k = 1, \ldots, n$

are linearly independent

0 - simplex $[v_0]$ $\cdot\, v_0$

1 - simplex $[v_0, v_1]$

2 - simplex $[v_0, v_1, v_2]$



3 - simplex $[v_0, v_1, v_2, v_3]$

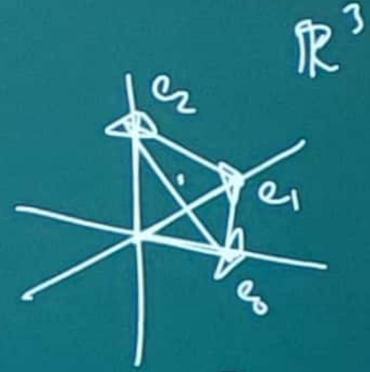
Defn n-simplex <u>ordered</u> collection $[v_0, v_1, \ldots, v_n]$ of vectors as before

↑ <u>important</u> <u>ordered</u> collection of vectors

in particular

$[v_0, v_1] \neq [v_1, v_0]$

<u>**Def^n**</u> the <u>standard</u> n-simplex is $[e_0, e_1, \ldots, e_n]$ in $\mathbb{R}^{n+1}$



$[e_0, e_1, e_2]$

<u>barycentric coordinates</u>

(t_0, t_1, t_2)

$$\boxed{\sum_{i=0}^{n} t_i = 1, \quad t_i \geq 0}$$

There is a canonical linear map

$$[v_0, v_1, \dots v_n] \to [w_0, w_1, \dots, w_n]$$

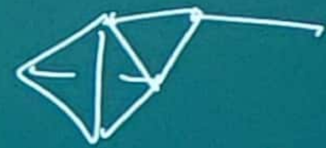
in particular

$$[e_0, e_1, e_2] \longrightarrow [v_0, v_1, v_2]$$

$$(t_i) \longmapsto \sum t_i v_i$$

this gives barycentric coordinates on any simplex $\sum t_i = 1 \quad t_i \geq 0$

<u>intuition</u>



<u>Def 2</u> A <u>face</u> of a simplex $[v_0, v_1, \dots, v_n]$ is the simplex corresponding to any subset of the vertices

Examples

faces ← preserve order of vertices

$[v_0]$ $\cdot v_0$

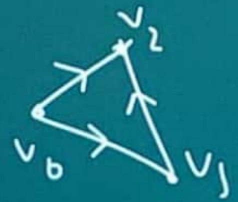
$[v_0]$ $\cdot$

$[v_0, v_1]$

$[v_0, v_1]$

$[v_0]$ $\cdot$ $[v_1]$ $\cdot$

$[v_0, v_1, v_2]$



$[v_0, v_1, v_2]$

$[v_1, v_2]$ $[v_0]$

$[v_0, v_1]$ $[v_1]$

$[v_0, v_2]$ $[v_2]$

$[v_0, v_1, v_2, v_3]$

1 3d-face

4 2d-faces

6 1-d faces

4 0- d-faces

$\underline{\text{Def}^{\underline{n}}}$ Simplicial complex

- simplices $\Delta_i^{n_i} = \{ \ldots, \overset{\Delta_1}{[v_0, v_1, \ldots, v_{n_1}]}, \ldots \}$
- face identifications $F = \{ \ldots, \overset{F_i}{\{[v_0, v_1, \ldots v_k], [w_0, \ldots w_k]\}}, \ldots \}$

geometric realization — topological space

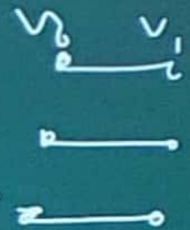
take $\underline{\bigsqcup \Delta_i / \sim \text{face pairings}}$

$\Delta = \{[v_0, v_1]\}$

$F = \{[v_0] \sim [v_1]\}$

S^1

$\underline{\text{Example}}$



$\Delta = \{ [v_0, v_1], [w_0, w_1], [u_0, u_1] \}$

$F = \{ [v_0] \sim [w_0], [w_0] \sim [u_0] \}$