Last time

Def^n Group G

operation $G \times G \longrightarrow G$
$(a,b) \longmapsto ab$

identity: $\forall a$ eae $ae = a$

inverse: $\forall g \in G\ \exists\, g^{-1}$ st $gg^{-1} = g^{-1}g = e$

associativity

Def^n G is abelian if $\forall a,b \in G$ $ab = ba$

notation

$ab \leftrightarrow a+b$ ⊛

Def^n $\alpha\ G \to H$ homomorphism if

$\alpha(ab) = \alpha(a)\alpha(b)$

⊛ $\alpha(a+b) = \alpha(a) + \alpha(b)$

Def^n $H \leq G$ subgroup

- $e \in H$
- $\forall h \in H$ $h^{-1} \in H$

$\forall h_1, h_2 \in H$, $h_1h_2 \in H$

Prop^n $\alpha\ G \to H$

$\alpha(G) \leq H$

$\ker(\alpha) \leq G$

$\{ g \in G \mid \alpha(g) = e_H \}$

Constructions (assume groups abelian)

· direct sum

$G \oplus H \leftarrow G \times H$

$\{ (g,h) \mid g \in G, h \in H \}$

operation

$(a,b)+(c,d) = (a+c, b+d)$

Ex $\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$

$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^3$

$\vdots$

quotient $H \leq G$

Def$^{\underline{n}}$ COSET cosets of H are the COSET sets

$a+H = \{ a+h \mid h \in H \}$

Example

$2\mathbb{Z} \subset \mathbb{Z}$

$\{ \ldots, -2, 0, 2, 4, \ldots \} \leq \mathbb{Z}$

$1+2\mathbb{Z}$ $\{ \ldots, -1, 1, 3, 5, \ldots \}$

Example

$3\mathbb{Z} \subset \mathbb{Z}$

$\uparrow \{ \ldots, -3, 0, 3, 6, \ldots \}$

$1 + 3\mathbb{Z} = \{ \ldots, -2, 1, 4, 7, \ldots \}$

$2 + 3\mathbb{Z} = \{ \ldots, -1, 2, 5, 8, \ldots \}$

Prop$^{\underline{n}}$ The cosets of $H \leq G$ form a <u>partition</u> of G

Proof define $a \sim b$ if

$b \in a + H$

check this is an equivalence relation

- reflexive $a \sim a$

  $0 \in H$ $\quad a = a + 0$

  $a \in a + H$

- symmetric $a \sim b \Rightarrow b \sim a$

$a \sim b \Rightarrow b \in a + H$

so $\exists h$ s.t. $b = a + h \Rightarrow a = b - h$

$\Rightarrow a \in b + H \Rightarrow b \sim a$

· transitive

$a \sim b,\ b \sim c \Rightarrow a \sim c$

$b \in a+H$ $\quad$ $c \in b+H$

$\exists h_1 \in H$ $\quad$ $\exists h_2$ s.t.

$b = a+h_1$ $\quad$ $c = b+h_2$

$c = a + \underbrace{h_1+h_2}_{\in H}$

so $c \in a+H$ ✓

$a \sim c$

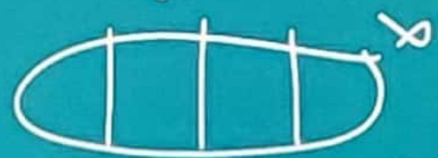
recall X set

$\sim$ equivalence relation

equivalence classes

$[x] = \{ y \in X \mid x \sim y \}$

this gives a partition of X



Conclusion $H \leq G$

cosets of H are a partition $\square$

$H \subseteq G$

cosets $a+H$

claim the cosets form a group with operation

$$(a+H)+(b+H) = (a+b)+H$$

check. well defined

$$a+h_1 + b+h_2 = a+b+\underbrace{(h_1+h_2)}_{\in H}$$

identity $0+H = H$

check

$a+H + H$

$$a+\underbrace{h_1 + h_2}_{\in H} = a+h_3 \Rightarrow a+H+H = H$$

· inverse $(a+H)^{-1} = -a+H$

check

$$a+h_1 + -a+h_2 = a-a+\underbrace{h_1+h_2}_{\in H}$$

$$a+H+(-a+H) = H$$

Examples $\qquad H \leq G$

$a + H$

- $2\mathbb{Z} < \mathbb{Z}$.

cosets

$0 + 2\mathbb{Z} = \{ \ ,-2,0,2, \ \}$ "0"

$1 + 2\mathbb{Z} = \{ \ ,-1,1,3, \ . \}$ "1"

$(1+2\mathbb{Z}) + (1+2\mathbb{Z}) = 2+2\mathbb{Z}$

$= 0+2\mathbb{Z}$

+	0	1
0	0	1
1	1	0

notation

$\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}_2$

- $3\mathbb{Z}$ $\mathbb{Z}$ $\qquad$ notation

$0 + 3\mathbb{Z}$

$1 + 3\mathbb{Z}$ $\qquad \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}_3$

$2 + 3\mathbb{Z}$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

in general $\mathbb{Z}/n\mathbb{Z} \leftrightarrow$ addition mod n

Example

$2\mathbb{Z} \oplus 3\mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z}$

$H \subset G$

$G \ni (a,b)$

cosets

$(a,b) + H$

$(a,b) + 2\mathbb{Z} \oplus 3\mathbb{Z}$

Exercise $G/H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$

Def$^{\underline{n}}$ A group G is cyclic if there is a $g \in G$ s.t. every element is $\underbrace{g+g+\dots+g}_{n \text{ times}} = ng$ for some $n \in \mathbb{Z}$

Examples $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$

1 1 1

Non-example $\mathbb{Z} \oplus \mathbb{Z}$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

$(1,0)$

$(0,1)$

Defn The order of G is $|G| = \#$ elements in G

Ex $|\mathbb{Z}_2| = 2$ $\quad \{0,1\}$

$|\mathbb{Z}_3| = 3$ $\quad \{0,1,2\}$

$|\mathbb{Z}| = \infty$

Defn $g \in G$ the order of g is the smallest n such that $ng = 0$

if no such n, say g is infinite order

Ex $\mathbb{Z}_2$ $\{0,1\}$ order 1, order 2 $\quad |e| = 0$

$\mathbb{Z} \oplus \mathbb{Z}_2$ $\quad (0,1)$ order 2

$(1,0)$ order ∞

Scanné avec CamScanner

$\underline{\text{Th}^{m}}$ Classification of finitely generated abelian groups

$\underline{\text{Def}^{n}}$ G is <u>finitely generated</u> if $\exists\ g_1, \dots, g_n$ st any $g \in G$ can be written as

$$g = a_1 g_1 + a_2 g_2 + \dots + a_n g_n$$

$$a_i \in \mathbb{Z}$$

$\leftarrow$ if G is abelian finitely generated then

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}_{d_1} \oplus \dots \oplus \mathbb{Z}_{d_n}$$

for some integers d_i

i.e. a direct sum of cyclic groups

recall $\mathbb{Z}^r = \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{r \text{ copies}}$

$$C_{d_1} = \mathbb{Z}/d_1\mathbb{Z} = \mathbb{Z}_{d_1}$$

Warning not true

for non-finitely generated

$\mathbb{Q}$ $\quad \frac{1}{a}, \frac{1}{b}$

contained in $\left\langle \frac{1}{ab} \right\rangle$

$\mathbb{Q}/\mathbb{Z}$ $\quad \frac{1}{a} \times a = 1$

Key observation

① $\mathbb{Z}_4 \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\{0,1,2,3\}$ $\quad \{(0,0),(1,0),(0,1),(1,1)\}$

order 4

every element is order ≤ 2

$(1,0)+(1,0)=(2,0)=(0,0)$

$(1,1)+(1,1)=(2,2)=(0,0)$

$\langle 1 \rangle \quad 1 \longmapsto (1,1)$

(2) $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$

$\{0, 1, 2, 3, 4, 5\}$ — 2: order 3, 3: order 2

$\{(0,0), (1,0), (0,1), (1,1), (0,2), (1,2)\}$ — $(1,0)$: order 2; $(1,1)$: order 6; $(0,1)$: order 3

$(1,1)+(1,1) = (2,2) = (0,2)$

$(0,2)+(1,1) = (1,3) = (1,0)$

$(1,0)+(1,1) = (2,1) = (0,1)$

$(0,1)+(1,1) = (1,2)$

$(1,2)+(1,1) = (2,3) = (0,0)$

$\mathbb{Z}$

2

$2+2 = 4$

$2+2+2 = 6 = 0 \pmod 6$

order 3

3

$3+3 = 6 = 0$

order 2

4

$4+4 = 8 = 2$

$4+2 = 6 = 0$

order 3

More generally

if p, q coprime

$$\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$$

$$\mathbb{Z}_{p^n} \not\cong \mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^b}$$

unless $a=0$ $b=n$

or $a=n$ $b=0$

Example

$$\mathbb{Z}_{24} \stackrel{?}{\cong} \mathbb{Z}_{12} \oplus \mathbb{Z}_2 \text{ ? No}$$

$$24 = 12 \times 2$$

$$3 \times 8 = \mathbb{Z}_3 \oplus \mathbb{Z}_8 \text{ Yes}$$

Scanné avec CamScanner

Maps between free abelian groups $\longleftrightarrow$ linear algebra over $\mathbb{Z}$ (no division)

homomorphisms

$$\alpha \quad \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$$

Observation α is a linear map

check

$$\alpha(x+y) = \alpha(x) + \alpha(y)$$

$$\alpha(ax) = a\,\alpha(x)$$

$$a \in \mathbb{Z}$$

$$ax = \underbrace{x + x + \cdots + x}_{a \text{ times}}$$

$$\alpha(x + \cdots + x) = \alpha(x) + \cdots + \alpha(x)$$

linear map α

$\longleftrightarrow$ matrix A

$m \left[\; n \; \right]$ $(m \times n)$-matrix

pick basis for $\mathbb{Z}^n, \mathbb{Z}^m$

ie a MINIMAL generating set

$\mathbb{Z}^n$

$(1, 0, 0, \dots, 0)$

$(0, 1, 0, \dots, 0)$

$\vdots$

$(0, 0, \dots, 1)$

$e_1 \longmapsto \alpha(e_1)$

$\alpha \quad \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$

basis $\quad$ basis

$e_1, e_2, \; , e_n \qquad f_1, \dots, f_m$

$x \in \mathbb{Z}^n$

$x = x_1 e_1 + x_2 e_2 + \cdot + x_n e_n$

$\alpha(x) = x_1 \alpha(e_1) + \cdots + x_n \alpha(e_n)$

$$\boxed{\alpha(e_1) = a_{11} f_1 + a_{21} f_2 + \cdots}$$

$\alpha(e_1) \leftarrow$

$$\boxed{\begin{aligned}\alpha(e_1) = {}& a_{11} f_1 \\ &+ a_{21} f_2 \\ &+ \\ & a_{m1} f_m\end{aligned}}$$

$$\begin{bmatrix} a_{11} & a_{12} & - \\ a_{21} & - & \\ - & & \end{bmatrix}$$

$$A = \begin{bmatrix} | & | & & | \\ \alpha(e_1) & \alpha(e_2) & \cdot & \alpha(e_n) \\ | & | & & | \end{bmatrix}$$