Algebraic Topology
and Applications

Joseph Ma

joseph.maher@csi.cuny.edu

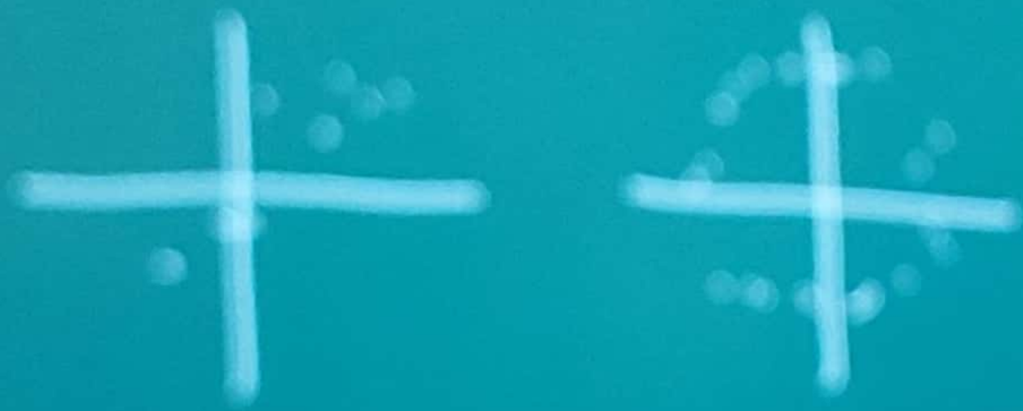
http://www math csi.cuny.edu/~maher/research

Topological data analysis

— persistent homology

Motivation

Data $X = \{x_i\}_{i=1}^{n} \in \mathbb{R}^d$



Algebraic Topology

Spaces

$\mathbb{R}$, $\mathbb{R}^n$

$S^1 \subset \mathbb{R}^2$

$x^2+y^2=1$

$S^n \subset \mathbb{R}^{n+1}$

surfaces

$M^n \leftarrow$ higher dimensional manifolds

simplicial complexes

Algebraic Invariants

- # path components

S^1  $\leftarrow$ 1 component

S^0 . . $\leftarrow$ 2 components

$\{-1,+1\}$

- fundamental group $\pi_1(X)$
- homology group $H_k(X)$

(spaces, continuous maps) $\longrightarrow$ (algebraic invariants: groups, homomorphisms)

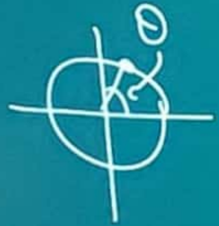
$f : X \longrightarrow Y$

$f_* : \pi_1 X \longrightarrow \pi_1 Y$

$f_* : H_k(X) \longrightarrow H_k(Y)$

$f : S^1 \longrightarrow S^1$

$f_* : \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$



$x^2+y^2=1$

$\theta \longmapsto 2\theta$

$n \longmapsto 2n$

(spaces, continuous maps) $\longrightarrow$ (algebraic invariants; groups, homomorphisms)

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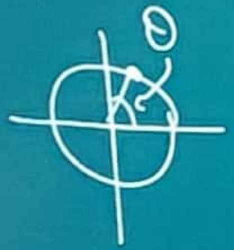

$x^2+y^2=1$

$\theta \longmapsto 2\theta$

Introduction to groups

Defn A group is a set G with a group operation

$$G \times G \longrightarrow G$$
$$(g,h) \longmapsto gh \quad | \quad g * h$$

properties 1) identity $e \in G$

for all $g \in G$ $\quad eg = ge = g$

2) inverses for all $g \in G$, there is g^{-1} such that

$$g g^{-1} = g^{-1} g = e$$

3) associativity

$g, h, k \in G$

$$(gh)k = g(hk)$$

Example $(\mathbb{Z}, +)$.

$$\mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$$
$$(a,b) \longmapsto a+b$$

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Examples

$(\mathbb{Z}, +)$ $a+b=b+a$

identity $0+a = a+0 = a$

inverses $a+(-a)=0$

$(\mathbb{R}, +)$

identity 0

inverse $-a$

$(\mathbb{R}\setminus\{0\}, \times)$ $\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$

$a, b \longmapsto ab$

$ab = ba$

identity 1

inverse $\frac{1}{x}$

We say G is <u>abelian</u> or <u>commutative</u>

if $gh = hg$ for all $g, h \in G$

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Examples

$(\mathbb{R}^n, +)$

identity $\underline{0} = (0, \ldots, 0)$

Inverse $-v$

$(M_{n,m}$ $(n \times m)$ matrices, addition$)$

Identity $0 = \begin{bmatrix} 0 & 0 & \cdots \\ & & 0 \end{bmatrix}$

Inverse $-A$

$\det(A) \neq 0$ $\rightarrow$ non-singular

$\big($ $(n \times n)$-matrices, multiplication$)$

2×2 identity I

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

inverse

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\uparrow$ not abelian

$AB \neq BA$

Examples

$(\mathbb{R}^n, +)$

identity $\underline{0} = (0, \dots, 0)$

Inverse $-v$

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↑ not abelian

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Examples

1

$f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous

additions $G \times G \longrightarrow G$

$f \quad g \qquad f+g \quad \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto f(x)+g(x)$

Identity $f(x) = 0$

inverse $-f(x)$

continuous bijections

2 $f \; \mathbb{R} \longrightarrow \mathbb{R}$

$G \times G \longrightarrow G$

$f \quad g \longmapsto f(g(x))$

identity $f(x) = x$

inverses $f^{-1}(x)$

$\uparrow$

$f(g(x)) \neq g(f(x))$

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Example

addition modulo n

$a+b \ (n)$

mod n

10 am + 3 hours
= 1pm.

$a+b \ (12)$

(24)

addition mod 3

$\{0, 1, 2\}$

↑

identity

$1+2=0$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Abelian

$\forall$ for all $g, h \in G$

$gh = hg$ $\quad AB = BA$

matrix group

2×2 matrices

$\exists$ there exists

$AB \neq BA$

Def^{n} A function $\phi\ G \to H$ is a homomorphism if

for all $a, b \in G$ $\quad \phi(ab) = \phi(a)\phi(b)$

Examples

① $\mathbb{Z} \xhookrightarrow{i} \mathbb{R}$

$a, b \qquad a, b$

$i(a+b) = i(a) + i(b)$

② $\mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \pmod 3$

$\phi(a+b) = \phi(a) + \phi(b)$

$a+b \pmod 3 = a \pmod 3 + b \pmod 3$

③ $\mathbb{Z} \xrightarrow[\phi]{\times d} \mathbb{Z}$

$n \longmapsto dn$

check

$d(n+m) = dn + dm$

$\phi(n+m) = \phi(n) + \phi(m)$

$\phi: G \to H$
homomorphism
useful check:
$\phi(\text{identity}) = \text{identity}$

<u>Def'n</u> $H \subseteq G$ a subset $H \subseteq G$
is a <u>subgroup</u> if H is a group
under the group operation for G

- suffices to check: $e \in H$
: if $h_1, h_2 \in H$ then $h_1 h_2 \in H$
: for all $h \in H$, $h^{-1} \in H$

<u>Examples</u> $(\mathbb{Z}, +) = G$

<u>subgps</u>

$(\{0\}, +)$ subgroup

$(2\mathbb{Z}, +)$ subgroup

$\{\ldots, -2, 0, 2, 4, \ldots\}$
[illegible]

Not (sub) groups

$\mathbb{N} \leq (\mathbb{Z}, +)$

$\uparrow$ no inverses

$(\mathbb{R}, \times)$

$\uparrow$ 0 doesn't have an inverse

Let $\phi : G \longrightarrow H$ be a homomorphism (e_G, e_H)

· image $\phi(G) \leq H$ subgroup

$\phi(G) = \{h \in H \mid \exists g \in G \text{ such that } \phi(g) = h\}$

Check

· $\phi(e_G) = e_H$

· inverses: $\phi(g)^{-1} = \phi(g^{-1})$

$\phi(gg^{-1}) = \phi(g)\phi(g^{-1})$

$\phi(e) = e$

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Example

$$(\mathbb{Z}, +) \xrightarrow{\phi} (\mathbb{R}\setminus\{0\}, \times)$$

$G \qquad H$

$$n \longmapsto 2^n$$

check

$$n+m \longmapsto 2^{n+m}$$

$$\phi(n+m) = \phi(n)\phi(m)$$

$$2^{n+m} = 2^n \times 2^m = 2^{n+m}$$

$$\phi(0) = 2^0 = 1$$

$\phi : G \longrightarrow H$ homomorphism

- $\phi(G) \leq H$ image(G)

subgroup

- kernel $\ker(\phi)$ subgroup

$$= \{g \in G \mid \phi(g) = e_H\}$$

e_G e_H identity

$\phi : G \longrightarrow H$

$\ker\phi = \{ g \in G \mid \phi(g) = e_H \}$

check subgp

$\phi(e_G) = e_H$, $e_G \in \ker(\phi)$

- spose $g \in \ker(\phi)$ means $\phi(g) = e_H$

$$\phi(g^{-1}) = \phi(g)^{-1} = e_H^{-1} = e_H$$

- spose $gh \in \ker(\phi)$ $\phi(gh) = \phi(g)\phi(h) = e_H e_H = e_H$ ✓

check

$ee = e$

$\Rightarrow e^{-1} = e$

$$\boxed{\begin{array}{l}\phi(G) \leq H \\ \ker(\phi) \leq G\end{array}}$$

Another method of constructing groups:

direct sum

$$G \oplus H$$

set $G \times H$

$$\{ (g,h) \mid g \in G, h \in H \}$$

operation $(a,b)(c,d) = (ac, bd)$

Examples

$\mathbb{Z} \oplus \mathbb{Z}$ $\quad (a,b) \quad a,b \in \mathbb{Z}$

$$(a,b) + (c,d) = (a+c, b+d)$$

notation

$$\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$$

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^3$$

$$(a,b,c) + (d,e,f) = (a+d, b+e, c+f)$$