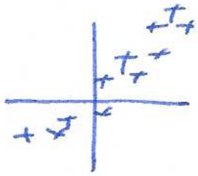
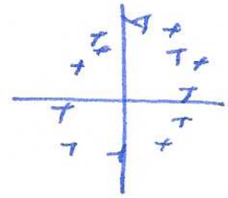
Algebraic topology and applications

Motivation (application) data, collection of points. $X = \{x_i\}_{i=1}^{n} \subseteq \mathbb{R}^{d}$ n, d large.

basic example ($X \subseteq \mathbb{R}^2$).



← look linear, try linear regression



← looks circular, try best fit circle.

general case $X \subseteq \mathbb{R}^d$ Q: can we find the shape, without knowing how to plot it?

A: yes, (sometimes) using persistent homology ← show example in R.

Aim of this course: cover mathematical background so we can understand output.

Algebraic topology: study topological spaces via algebraic invariants.

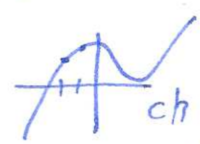
spaces: $\mathbb{R}^n$, S^1, S^n, surface ☺ S^2, (torus), (double torus), ...

invariants: # of connected components, Euler characteristic, fundamental group, homology...

Topological spaces and continuous maps

recall: a map $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x \in X$ if for all $\epsilon > 0$ there is a $\delta > 0$ s.t. if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

$\mathbb{R} \xrightarrow{f} \mathbb{R}$

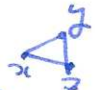


cts

not cts.

Q: how can we generalize this to more general spaces?

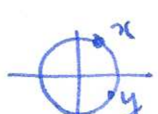
Def$^{\underline{n}}$ (X, d) is a metric space if X is a set, and $d: X \times X \to \mathbb{R}_{\geq 0}$ is a distance function, i.e.
- for all $x \in X$, $d(x,x) = 0$
- for all $x, y \in X$, $d(x,y) = d(y,x)$
- for all $x,y,z \in X$, $d(x,y) + d(y,z) \geq d(x,z)$ (triangle inequality)



Examples ① $(\mathbb{R}^n, d_E)$ Euclidean metric $d(x,y) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n - y_n)^2}$.

② any subset of a metric space w/ restriction metric.

take $S^1 \subseteq \mathbb{R}^2$, $x^2 + y^2 = 1$ warning this is not the angular metric on S^1!

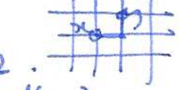


$d_{\mathbb{R}^2}(x,y) = \sqrt{2}$ $d_\theta(x,y) = \frac{\pi}{2}$

(3) any graph with the induced path metric in which every edge has length 1.

<u>Def</u>: A <u>graph</u> X consists of: a set of vertices $\{x_0, x_1, \dots\}$.
a set of edges $\{(x_i, x_j), \dots\}$.

<u>Examples</u> 

<u>Induced path metric</u> · for each edge (x_i, x_j) identify this with unit interval $[0,1]$ with Euclidean metric $\underset{s\ t}{\cdot\ \cdot}$ $d(s,t) = |t-s|$.
· for any two points x, y, consider all paths from x to y, and choose shortest one. 
$d(x,y) = 2$.

<u>Def</u>: A map $f : (X, d_X) \to (Y, d_Y)$ is <u>continuous</u> at x if for all $\epsilon > 0$ there is a $\delta > 0$ such that $\forall y$ if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Q: how can we generalise this to more general spaces?

<u>Def</u>: A topological space X (or (X,T)) is a set X, together with a ~~sets~~ collection of subsets $T \subseteq P(X)$ ← power set of X, i.e. all subsets, called the <u>open</u> sets, with the following properties: · $\emptyset, X \in T$
· finite intersections: if $U_1, \dots, U_n \in T$, then $U_1 \cap \dots \cap U_n \in T$.
· arbitrary unions: if $X_\alpha \in T$ $\alpha \in A$. then $\bigcup_{\alpha \in A} X_\alpha \in T$.

<u>Def</u>: A map $f : X \to Y$ (or $f : (X, T_X) \to (Y, T_Y)$) is <u>continuous</u> if for all open $U \in T_Y$ (or open $U \subseteq Y$) $f^{-1}(U)$ is open in X, ie $f^{-1}(U) \in T_X$
$f^{-1}(U) \subseteq X$ open.

<u>Def</u>: Let (X,T) be a topological space $B \subseteq T$ is <u>a basis</u> for for the topology, if every $U \in T$ is a union of sets in B.

<u>Example</u> (1) Topology on $\mathbb{R}$: choose $B =$ set of all open intervals (a,b). 
so $(\mathbb{R}, T)$ $T =$ sets which are (arbitrary) unions of open intervals

<u>consequence</u> if $U \subseteq \mathbb{R}$ is open and $x \in U$, then there is an open interval $x \in (a,b) \subseteq U$.

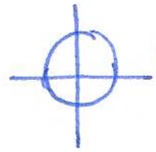
(2) Topology on (X, d_X) set $B =$ open balls $B(x,r) = \{y \in X \mid d_X(x,y) < r\}$.

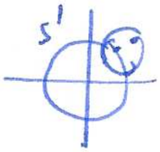
<u>note</u>. this gives <u>same</u> topology on $\mathbb{R}$ with standard metric $d_X(x,y) = |x-y|$.

- $f: (X, d_X) \to (Y, d_Y)$ is cts with metric def^n iff cts w/ topological definition.

(3) restriction / subset topology $Y \subseteq (X, T_X)$ <u>Def^n</u> $U \subseteq Y$ is open if there is an open set $V \subseteq X$ s.t. $U = X \cap V$.

<u>Example</u> $S^1 \subseteq \mathbb{R}^2$
$\mathbb{R} \subseteq \mathbb{R}$





(4) quotient topology. (X, T_X) topological space, $\sim$ an equivalence relation on X set $\overline{X} = X/\sim$ set of equivalence classes. ~~Define $T_{\overline{X}}$ by~~ Let $q: X \to \overline{X}$ be the map that sends $x \mapsto [x]$. <u>Def^n</u> $U \in T_{\overline{X}}$ is open if $q^{-1}(U)$ is open in X.

<u>Example</u> unit interval $I = [0,1]$

$0 \sim 1$
$x \sim x$ for all x.

<u>claim</u> $I/\sim = S^1$.

<u>Example</u>: $\mathbb{RP}^n = S^n/$antipodal map. • Cone CX • suspension SX.

(5) • <u>discrete topology</u> (X, T_X) where $T_X = P(X)$ ← natural topology on discrete set of points $\therefore \subseteq \mathbb{R}^n$.

• <u>indiscrete topology</u> $(X, \{\phi, X\})$.

Q: when are two topological spaces equivalent?

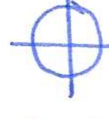
<u>Def^n</u> X, Y are <u>homeomorphic</u> if there is a bijection $f: X \to Y$ s.t. both f and f^{-1} are continuous.

<u>Warning</u> need both f, f^{-1} cts, example $f: \underset{\text{discrete}}{(X, P(X))} \to \underset{\text{indiscrete}}{(X, \{\phi, X\})}$.
$f = id_X$.

f cts, but f^{-1} <u>not</u> cts.

<u>Observation</u> when is $f: X \to Y$ cts? suffices to pick basis set open $B \subseteq Y$ and show for any $x \in f^{-1}(B)$ there is a basis set $A \subseteq X$ s.t. $x \in A \subseteq f^{-1}(B)$.

<u>Example</u> S^1:



- restriction topology
- topology from restriction metric
- topology from angular metric

} <u>Exercise</u> show these are all homeomorphic.

- $I = [0,1]/\sim$ $\;\; \begin{matrix} 0 \sim 1 \\ x \sim x \end{matrix}$ ← this is also S^1. • $S^1/$antipodal map.

<u>More examples</u> • <u>product topology</u> on $X\times Y$ ← as a set just cartesian product

$X\times Y = \{(x,y) \mid x\in X, y\in Y\}$ A basis for the product topology is given by all sets of the form $U\times V$, U open in X, V open in Y. Suffices to choose U,V to be basis sets for bases of U and V.

<u>Exercise</u> • show the product topology on $\mathbb{R}^2=\mathbb{R}\times\mathbb{R}$ is the same as the topology induced by the standard Euclidean metric.

<u>Exercise</u> • show the restriction topology $\mathbb{R}\subset\mathbb{R}\times\mathbb{R}$ is the same as the original topology.

• show the projection maps $p_X: X\times Y\to X$, $p_Y: X\times Y\to Y$ are continuous.

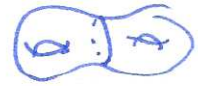
<u>Quotients</u> • Let $S^n\subset\mathbb{R}^{n+1}$, there is an antipodal map $\underline{x}\mapsto -\underline{x}$. The quotient is called $\mathbb{RP}^n$. Draw pictures for $n=0,1,2$.

• Wedge products/sums. $(X,x_0)\vee(Y,y_0)$ is $X\sqcup Y/\sim$ where the equivalence relation is: $x_0\sim y_0$, all other points just equivalent to themselves.

<u>Examples</u> draw pictures of $S^1\vee S^1$, $S^2\vee S^1$, $S^2\vee S^1$, $(S^1\times S^1)\vee S^1$, $(S^1\times S^1)\vee(S^1\times S^1)$ and some glue sets.

• Gluings: Let $A\subset X$ and $B\subset Y$, and $f:A\to B$. Then let Z be ~~X~~ X and Y glued together by f, i.e. $X\sqcup Y/\sim$ equivalence relation is $a\sim f(a)$ for all $a\in A$, ~~no other eq~~ all other equivalence classes just consist of one point.

<u>Examples</u> take $I=[0,1]$, two copies I' and glue 0 to $0'$. – show result is homeo to $[0,1]$. EX

take: $X_1 = X_2 =$  and glue circles together: 

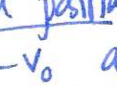
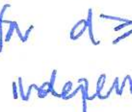
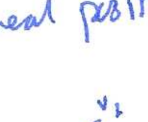
• Self gluings, can choose $A,B\subset X$ and $f:A\to B$, can construct $X/\sim$ where equivalence relation is $a\sim f(a)$, no other equivalences except self equivalences.

<u>Examples</u>: $[0,1]/_{0\sim 1}$. $S^1\times I$, $f: S^1\times 0\to S^1\times 1$ by identity map ⇝ 

• crush subsets: $A\subseteq X$. construct $X/A = X/\sim$ where for all $a,b\in A$, $a\sim b$.

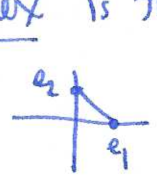
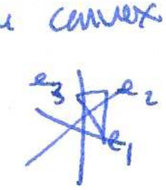
<u>Example</u> $X=B^n$, $A=\partial B^n = S^{n-1}$ $B^n/\partial B^n\cong S^n$.

<u>Simplicial complexes</u>. <u>Defn</u> an n-simplex $[v_0,\dots v_n]$ is the convex hull of a collection of $(n+1)$-points in <u>general position</u> in $\mathbb{R}^d$ for $d\geq n+1$. General position means that the n vectors $v_i - v_0$ are linearly independent.

<u>Examples</u> $[v_0]$. $[v_0,v_1]$.  $[v_0,v_1,v_2]$  $[v_0,v_1,v_2,v_3]$  ...

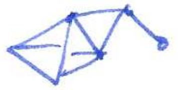
<u>Important convention</u> we remember the order of the vertices so $[v_0,v_1]\neq[v_1,v_0]$.

<u>Defn</u> the <u>faces</u> of $[v_0,\dots v_n]$ are just spans of subsets $[v_{i_1},\dots v_{i_k}]$.

__Def__: the standard __n-simplex__ is the convex hull of the standard unit vectors $e_1, \dots e_{n+1}$ in $\mathbb{R}^{n+1}$. Eg:   etc.

we have __preferred coordinates__ called __barycentric coordinates__ $(t_0, t_1, \dots, t_n)$ which satisfy: $\sum_{i=0}^{n} t_i = 1$ and $t_i \geq 0$ for all i.

__Intuition__: a __simplicial complex__ is constructed from a disjoint union of simplices by identifying their __faces__ = subsimplices.

__Example__  __key property__ the interior of each (sub) simplex is embedded in X.

__problem__ $\begin{matrix} v_1 \\ v_0 \end{matrix} \leftrightarrow \begin{matrix} w_0 \\ w_1 \end{matrix} \leftrightarrow \begin{matrix} v_0 \\ v_1 \end{matrix}$ ← this sequence of gluings folds the edge in half.

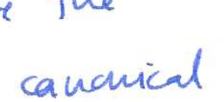
__solution__ whenever we glue two faces $[v_0, \dots v_n] \sim [w_0, \dots w_n]$ we will use the canonical linear map that preserves the order of the vertices.

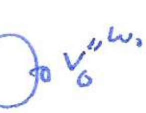
so $[v_0, v_1] \sim [w_0, w_1]$ allowed $\quad [v_0, v_1] \sim [w_1, w_0]$ not allowed.

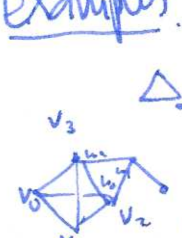
__Def__ A ~~simplicial complex~~ Δ-complex ~~is a disjoint~~ consists of the following data:

- a list of simplices $\Delta_1^{n_1}, \Delta_2^{n_2}, \dots \quad \{\Delta_\alpha^{n_\alpha} \mid \alpha \in A\}$.
- a list of face pairings. $\{F_\beta \mid \beta \in B\}$, where each pairing F_β consists of two faces of the same dimension. Note: the same face may occur in more than one pairing.

The __geometric realization__ of the Δ-complex is the topological space X formed from the disjoint union of the simplices $\bigsqcup \Delta_\alpha^{n_\alpha} / \sim$ where the equivalence is given by gluing the faces in F_β together according to the canonical linear maps.

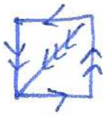
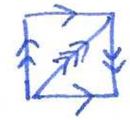
__Examples__. $\{[v_0, v_1], [w_0, w_1]\} = \Delta_\alpha \quad F_\beta = \{\{[v_0], [w_0]\}, \{[v_1], [w_1]\}\}$ 

$\Delta_\alpha = \{[v_0, v_1], [w_0, w_1]\} \quad F = \{\{[v_0], [v_1]\}, \{[v_0], [w_0]\}, \{[w_0], [w_1]\}\}$ 

 $\Delta = \{[v_0, v_1, v_2, v_3]$, $[w_0, w_1, w_2]$, $[u_0, u_1]\}$. $F = \{\{[v_2, v_3], [w_0, w_2]\}, \{[w_1], [u_0]\}\}$..

Examples $S^1 \times S^1$  $\Delta = \{[v_0, v_1, v_2], [w_0, w_1, w_2]\}$

$F = \{[v_0, v_1] \sim [w_1, w_2], [v_0, v_2] \sim [w_0, w_2], [v_1, v_2] \sim [w_0, w_1]\}$.

$\mathbb{RP}^2$  ! not Δ-complex. K  $\mathbb{RP}^2$:  ← works

More general construction cell / CW-complexes.

X^0 ← discrete collection of points

X^1 ← take disjoint union of 1-balls B^1_α and gluing maps $\varphi_\alpha : \partial B^1_\alpha \to X^0$.

X^2 ← take disjoint union of 2-balls  B^2_α and gluing maps $\varphi_\alpha : \partial B^2_\alpha \to X^1$.

etc. more flexible, but requires more work to deal with.

Examples S^2  $T^2 = S^1 \times S^1$   glue on disk.

Useful properties of topological spaces

Hausdorff: for all distinct $x, y \in X$, there are open sets U, V s.t. $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Separable: X contains a countable dense set.

Compact: every open cover of X has a finite subcover. eg. $[0,1]$ ← compact, $(0,1)$ ← not compact.

Homotopy recall, X, Y homeomorphic if $\exists$ bijection $f : X \to Y$ s.t. f, f^{-1} cts.

Problem: classify topological spaces up to homeomorphism ← too hard!

Look for weaker notion of equivalence coming from continuous deformations.

Consider: $f : X \to Y$ what might we mean by a continuously varying family of maps $f_t : X \to Y$.? Defn A homotopy is a cts. map $F : X \times I \to Y$,

notation write $f_t(x) = F(x, t)$.

Example $X = \{pt\}$. $f : \{x\} \to X$.

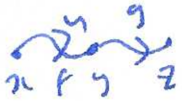
$F : \{x\} \times I \to X$. ← i.e. a path.

Defn A path is a cts. map $f : I \to X$.

Defn Define an equivalence relation on X by $x \sim y$ if there is a path $f : I \to X$ s.t. $f_0(x) = x$ and $f_1(x) = y$. Ex check this is an equivalence relation.

- reflexive: Let $f: I \to x \in X$, this is a path from x to x so $x \sim x$.
- symmetric: given $f: I \to X$, a path from $x = f(0)$ to $y = f(1)$
  define $\bar{f}: I \to X$ by $\bar{f}(t) = f(1-t)$, the <u>reverse path</u>, is a path from $\bar{f}(0) = y$ to $\bar{f}(1) = x$.

- transitive: Let f be a path from x to y, and g be a path from y to z



define $f \cdot g: I \to X$ by $f \cdot g(t) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$

this is a path from x to z, so $x \sim y$ and $y \sim z \Rightarrow x \sim z$

<u>Def$^{\underline{n}}$</u> The <u>path components</u> of X are the equivalence classes under this relation.

<u>Remark</u> we have defined our first topological invariant: top spaces $\to \mathbb{N}$.
$X \longmapsto$ #path components of X.

<u>Observation</u> if X homeo Y then #path components of X = #path components of Y.

~~Prop^n~~ Really $X = \bigsqcup X_i$ path components, ~~X_i open and closed~~ $\cancel{\text{}}$.

<u>Prop^n</u> $S^0 = \{x\} \sqcup \{y\}$ has two path components.

<u>Proof</u> suppose only one, then there is a path $f: I \to S^0$ s.t. $f(0) = x$ and $f(1) = y$
note $\{x\}, \{y\}$ are open in S^0 (in fact an open cover) so $f^{-1}(\{x\})$ and $f^{-1}(\{y\})$ are open in I, and in fact an open cover. Note $0 \in f^{-1}(\{x\})$ and $1 \in f^{-1}(\{y\})$.
$1 \notin \qquad 0 \notin$.
$f^{-1}(x)$ is a non-empty subset of $[0,1] = I$, contains 0, but not 1, has a supremum = least upper bound s. Recall, for all $\epsilon > 0$, $\exists\ t \in f^{-1}(\{x\})$ s.t. $t \in [s-\epsilon, s]$.

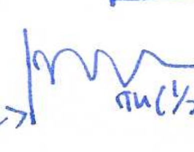
- suppose $s \in f^{-1}(\{x\}) \leftarrow$ open, so there is a basis set / open interval $(a,b) \subseteq f^{-1}(\{x\})$ s.t. $s \in (a,b) \subseteq f^{-1}(\{x\})$ but then $\exists\ s+\epsilon > s$ in $f^{-1}(\{x\})$ # so $s \notin f^{-1}(\{x\})$.
- therefore $s \in f^{-1}(\{y\}) \leftarrow$ open, so $\exists$ basis set / open interval s.t. $s \in (a,b) \subseteq f^{-1}(\{y\})$, but then $s - \epsilon$ is bigger than all elements of $f^{-1}(\{x\})$ so s not sup #. $\square$.

<u>Remark</u> we have shown that $I = [0,1]$ is not the disjoint union of two open sets.

<u>Def$^{\underline{n}}$</u> X is <u>disconnected</u> if $X = U \sqcup V$, U, V both open.

<u>Connected</u> vs <u>path connected</u> $\leftarrow$ not the same!

Example.



$\leftarrow$ this is <u>connected</u> but not <u>path connected</u>.

<u>Remark</u> no difference for Δ-complexes, cell complexes.

Application $\mathbb{R}^1 \not\cong \mathbb{R}^2$ ← note: there is a bijection from $\mathbb{R}^1$ to $\mathbb{R}^2$
there is an onto cts map from $\mathbb{R}^1$ to $\mathbb{R}^2$.

Proof suppose $f:\mathbb{R}^1\to\mathbb{R}^2$ is a homeomorphism. then this gives a homeomorphism

$f: \mathbb{R}^1\setminus\{0\} \to \mathbb{R}^2\setminus f(0)$.  ← one connected component. $\#$ □

↑ two connected components

Aim: generalise this argument to higher dimensions (homology) ...

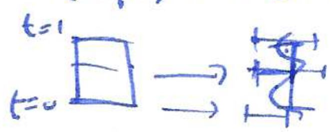
Topological spaces up to homotopy

Defn we say X, Y are homotopy equivalent $X\simeq Y$ if there are maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$
such that $gf\simeq \mathbb{1}_X$ and $gf\simeq \mathbb{1}_Y$.

Special case X is contractible if $X\simeq\{x\}$ point

Example $I=[0,1]$ is contractible. define $F: I\times I \to I$.
$(x,t)\mapsto (1-t)x$

then $f_0(x) = F(x,0) = x = \mathbb{1}_I$.
$f_1(x) = F(x,1) = 0$



Exercises show I^n, $\mathbb{R}^n$, B^n contractible.

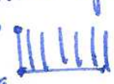
(non)-example: $S^0 = \{x\}\sqcup\{y\}$ ← not contractible.

Q: how do we show S^1 not contractible, or any other space for that matter?

variant: retractions. Let $A\subset X$ then X retracts to A is there is a map
$A \underset{i}{\overset{r}{\leftrightarrows}} X$ s.t. $r|_A = \mathbb{1}_A$

we've shown: $[0,1]$ retracts to $[0]$.

Bad example there is a contractible space X which does not retract to any point

Not so bad example comb space  ← retracts to a but not to b.

Q: how does # of path components behave under cts maps?

Propn $f: X\to Y$ cts, then # of path components of $f(X)$ ≤ # of path components of X.

Proof if $\gamma: I\to X$ is a path from $\gamma(0)$ to $\gamma(1)$, then $f\gamma: I\to Y$ is a path from $f\gamma(0)$ to $f\gamma(1)$
$f(x)$ to $f(y)$. □

Prop$^{\underline{n}}$ If X is contractible, then # path components of $X = 1$.

Proof there is a contraction $F: X \times I \to X$ s.t. $f_0 = \mathbb{1}_X$, $f_1(x) = x_0 \in X$.

pick $y \in X$, then $f_t(y) = F(y,t)$ is a path from y to x_0. $\square$.

more generally: **Prop$^{\underline{n}}$** If $X \simeq Y$ homotopy equivalent, then they have the same number of path components.

Proof $X \underset{g}{\overset{f}{\rightleftarrows}} Y$. suppose $\overset{a,b}{\cancel{x, y}} \in X$ lie in the same path component, then there is a path $\gamma: I \to X$ s.t. $\gamma(0) = a$, $\gamma(1) = b$, so $f\gamma$ is a path from $f(a)$ to $f(b)$ in Y. so $f(x_i)$ lies in a single path component of Y. Now spose $a, b \in X$ lie in different path components, but $f(a), f(b)$ lie in ~~diff~~ same path component of Y. (↑ same path component) Then $\exists\, \gamma: I \to Y$ s.t. $\gamma(0) = f(a)$ and $\gamma(1) = f(b)$. The $g\gamma: I \to X$ is a path from $\underset{\neq a}{g\gamma(0)}$ to $\underset{\neq b}{g\gamma(1)}$.

i.e. a path fr $gf(a)$ to $gf(b)$. But $gf \simeq \mathbb{1}_X$ so $\exists\, H: X \times I \to X$ s.t. $h_0(x) = x$ and $h_1(x) = gf(x)$, so $h_t(a)$ is a path from a to $gf(a)$, $h_t(b)$ is a path from b to $gf(b)$ so $a \sim gf(a)$, $b \sim gf(b)$

$\Rightarrow a, b$ lie in same path component ※ $\square$.

Fundamental group ← invariant of (path connected) spaces, will help us distinguish more spaces. (X, x_0) ← space w/ basepoint. $\rightsquigarrow$ $\pi_1(X, x_0)$ ← group.

Groups (quick review). **Examples** $(\mathbb{Z}, +)$. $\left(\begin{smallmatrix}\text{non-sing}\\ \text{matrices}\end{smallmatrix}, \begin{smallmatrix}\text{matrix}\\ \text{mult}\end{smallmatrix}\right)$. (permutations).

Def$^{\underline{n}}$ A group G is a set G, together with a multiplication $G \times G \to G$ with the following properties: • identity: $\exists\, 1 \in G$ s.t. $1g = g1$ for all $g \in G$

• inverses: for all $g \in G$, there is $g^{-1} \in G$ s.t. $gg^{-1} = 1 = gg^{-1}$.

• associativity: for all $g, h, k \in G$ $(gh)k = g(hk)$.

Examples
- $(\mathbb{Z}, +)$ identity 0, inverse of n is $-n$
- $(\mathbb{R}, +)$ • $(\mathbb{R} \setminus \{0\}, \times)$ identity 1, inverse of x is $\frac{1}{x}$.
- $SL(2, \mathbb{Z})$ ← note $AB \neq BA$ in general, this is a noncommutative / non-abelian group.
- F_2 = free group on two generators: $\langle a, b \mid \ \rangle$ elements: all finite strings in $\{a^{\pm 1}, b^{\pm 1}\}$.

up to equivalence
$uaa^{-1}v \sim uv$
$ua^{-1}av \sim uv$
$ubb^{-1}v \sim uv$ etc.

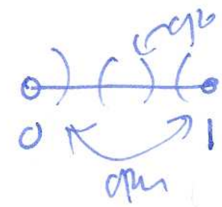
identity element $1 = \phi$.
multiplication: concatenation of strings.

relation on X, then quotient top on $\overline{X} = X/\sim$ is:

$X \xrightarrow{p} X/\sim = \overline{X}$ $\quad U \subseteq \overline{X}$ is open iff $p^{-1}(U)$ is open in X.

<u>Example</u> $X = [0,1] \subseteq \mathbb{R}$ induced top.

$\sim$: $0 \sim 1$, $x \sim x$



$\overline{X} \cong S^1$.

<u>Example</u> Δ-complex.

<u>Example</u> CW-complex.

<u>Groups</u> <u>motivation</u> $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{C}$, matrices: sets, w/ operations $+$, $\times$..

<u>Defn</u> A <u>group</u> G is a set G together with a multiplication $G \times G \to G$ with the following properties:

identity: $\exists\, 1 \in G$ s.t. $1g = g1$ $\forall g \in G$.

inverse: $\forall g \in G$ $\exists\, g^{-1} \in G$ s.t. $g^{-1}g = gg^{-1} = 1$.

associativity: $\forall g, h, k \in G$, $(gh)k = g(hk)$.

<u>Examples</u> $(\mathbb{Z},+)$, $(\mathbb{R},+)$, $(\mathbb{C},+)$, ~~n×n~~ ~~invertible~~ n×m matrices, matrix addition

<u>Not example</u> $(\mathbb{N},+)$ $(\mathbb{R},\times)$.

<u>Examples</u> $(\mathbb{R}\setminus\{0\},\times)$, $(\mathbb{C}\setminus\{0\},\times)$, $n\times n$ invertible matrices, matrix multiplication

<u>Remark</u>. for matrices $AB \neq BA$ in general.

<u>Example</u> Free group F_n. <u>Defn</u> subgp $H \subseteq G$ subset s.t. if $h \in H$ then $h^{-1} \in H$.

Defn A map $f: G \to H$ between two groups is a homomorphism if $\forall$ ~~g~~ $g_1, g_2 \in G$, $f(g_1 g_2) = f(g_1) f(g_2)$.

~~Defn~~ **Defn** The kernel of f is ~~{~~ $\ker(f) = \{ g \in G \mid f(g) = 1_H \in H \}$

Propn $\ker(f)$ is a subgroup of G.

Quotient groups : special case: abelian groups

Defn ~~gr~~ G is a *abelian* if $gh = hg$ $\forall g, h \in G$.

warning often write abelian gps *additively*.

Let $H \subseteq G$ abelian, the cosets of H are the sets $gH \subseteq G$

claim cosets form a *partition* of G.

claim cosets $\{gH\}$ form a group, with group operation

$g_1 H \, g_2 H = (g_1 g_2) H$ ← check well defined

check id: $1H = H$.

inverse $(gH)^{-1} = g^{-1} H$.

associativity ✓.

Examples

$0 \in (\mathbb{Z}, +)$. $\mathbb{Z}/0 = \mathbb{Z}$

$\mathbb{Z} \subseteq \mathbb{Z}$ $\mathbb{Z}/\mathbb{Z} = 0$.

$2\mathbb{Z} \subseteq \mathbb{Z}$ $\mathbb{Z}/2\mathbb{Z}$ = abelian gp w/ 2 elements.

$n\mathbb{Z} \subseteq \mathbb{Z}$ $\mathbb{Z}/n\mathbb{Z}$ = cyclic group w/ n elements.

General (non-abelian) case

<u>Defn</u> $N \leq G$ is a <u>normal</u> subgp if $gN = Ng$.

<u>Defn</u> G/N = cosets gN with group op $(gN)(hN) = ghN$.

<u>Defn</u> A finitely presented group is $\langle g_1, g_2, \ldots \mid r_1, r_2 \ldots r_\ell \rangle$

r_i words in g_i. take F = free gp generated by g_i

quotient by $N = \langle\langle r_i \rangle\rangle$ normal subgp generated by r_i

$G = F/N$.

<u>Examples</u> $\mathbb{Z} = \langle a \mid \rangle$. $\{ \ldots a^{-2}, a^{-1}, 1, a, a^2, \ldots \}$.

$\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z} = \langle a, b \mid ab = ba \rangle$ $\{ a^k b^\ell \}$. $a^k b^\ell \cdot a^i b^j = a^{k+i} b^{\ell+j}$.

$\mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n = 1 \rangle$.

$F_n = \langle g_1 \ldots g_n \mid \rangle$.

<u>Problem</u> ① Q: what is $SL(2,\mathbb{Z})$? A: $\langle A, B \mid A^4 = B^6 = \rangle$.

② Q: is $\langle g_1 \ldots g_n \mid r_1 \ldots r_\ell \rangle$ the trivial group

<u>Thm</u> [Novikov] no algorithm to decide this.

We'll (mainly) look at invariants with values in abelian groups

<u>Thm</u> Classification of finitely generated abelian gps

$G \cong \mathbb{Z}^r \oplus \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_n}$ p_i not nec distinct prime powers.

<u>Key fact</u> $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$ but $\mathbb{Z}_4 \not\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Homotopy: motivation: continuous deformations

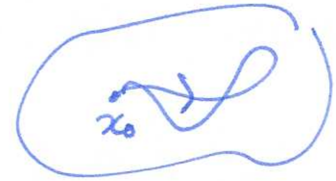
Def^n A homotopy is a cts. map $F: X \times I \to Y$.

think: $f_t(x) = F(x,t)$ then f_t is a continuously changing family of maps. } homotopy equivalence

Fundamental group ← first example of algebraic invariant.

$X \rightsquigarrow \pi_1 X$

top sp (not nec abelian) group



intuition: pick base pt $x_0 \in X$.

consider loop based at x_0, i.e. map $\gamma: I \to X$ s.t. $\gamma(0) = \gamma(1) = x_0$.

compositions γ_1, γ_2 define $\gamma_1\gamma_2 : I \to X$

Q: can we make this a group?

identity: $1 : I \to x_0$. problem $\bar{\gamma}\gamma \neq 1$!

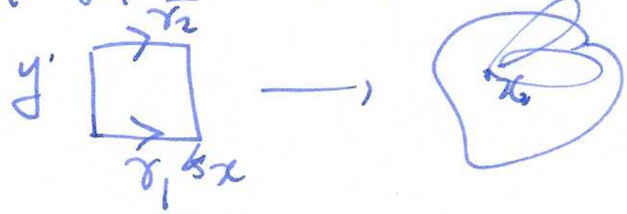
inverse: $\bar{\gamma} : I \to X$ $\bar{\gamma}(t) = \gamma(1-t)$

not (associativity)?! $(\gamma_1\gamma_2)\gamma_3$ $\gamma_1(\gamma_2\gamma_3)$.

but there are homotopic maps (in fact same up to reparametrization).

Fix consider loops, up to based homotopy.

we say $\gamma_1 \sim \gamma_2$ if $\exists\ H: I \times I \longrightarrow X$.

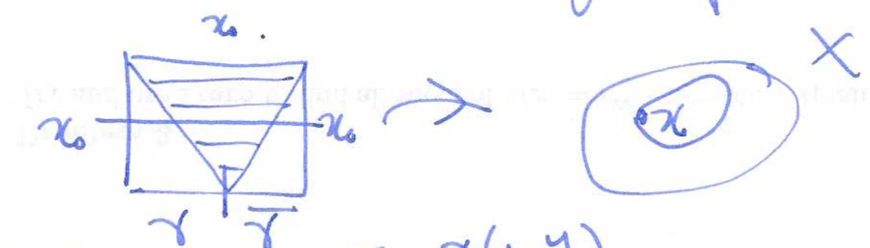


s.t. $H(x,0) = \gamma_1(x)$
$H(x,1) = \gamma_2(x)$
$H(0,y) = x_0 = H(1,y)$

Defn $\pi_1(X, x_0) \Leftarrow$ based loops up to homotopy, group composition given by path composition of loops

Lemma: check this is a group.

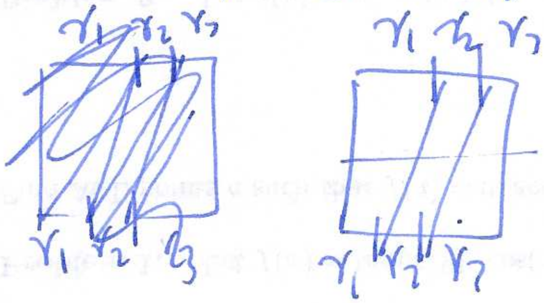
inverses:



idea: $H(x,y) =$ $\gamma(1-\frac{y}{2})$.

$\gamma|_{[0,1-\frac{y}{2}]}$ $\bar{\gamma}|_{[1-\frac{y}{2},0]}$

associativity

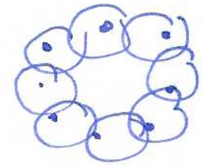


Example $\{pt\}$ $\mathbb{R}^n$ $\pi_1 = \{1\}$. $\leftarrow$ any contractible space.

$\pi_1(S^1) = \mathbb{Z}$. $\pi_1(T^2 = S^1 \times S^1) = \mathbb{Z}^2$

$\pi_1(S^2) = \{1\}$ $\pi_1 = \langle a,b,c,d \mid [a,b] = [c,d] \rangle$

Next invariant : simplicial homology.



Data set : point cloud $X = \{x_i\} \subseteq \mathbb{R}^n$

want : topological space. idea : replace x_i with $B_r(x_i)$

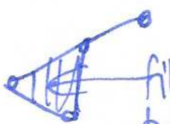
problems ① computing intersection of balls hard in high dim

② what should we choose for r?

①

Defn Vietoris–Rips complex : $VR_\epsilon(X)$.

- vertices, vertices $X = \{x_i\}$.
- edges : connect x_i, x_j if $d(x_i, x_j) < 2r$
- fill in simplices.



Q : why do this? Compare with Čech complex :

Defn $C_\epsilon(X)$ has vertices : . points in X

. k-simplex $[x_0, x_1 \dots x_k]$ if $\bigcap_{i=0}^{k} B_\epsilon(x_i) \neq \phi$.

This is the 'reverse' operation of the nerve of a cover.

Let $\{U_i\}$ be a open cover of X, then $N(\{U_i\})$ is a simplicial complex with

- vertices : corresponding to sets U_i
- k-simplices : $[i_0, \dots i_k]$ if $\bigcap_{j=0}^{k} U_{i_j} \neq \phi$.

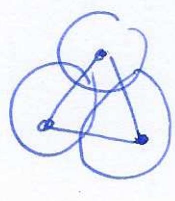
Thm If $\{U_i\}$ is an open cover of X s.t. all non-empty intersections $\bigcap U_{i_j}$ are contractible, then $N(\{U_i\}) \simeq X$.

Corollary $\bigcup_{x \in X} B_\epsilon(x) \underset{\text{homeo}}{\cong} |C_\epsilon(X, d_X)|$.

Fact $\quad C_\epsilon(X) \subseteq VR_\epsilon(X) \subseteq C_{2\epsilon}(X)$.

So $VR_\epsilon(X)$ not nec. homeo. to $\bigcup B_\epsilon(X)$ but seems good enough geometric approx in practice.

Example



→ Rips contains triangle

↳ Čech doesn't.

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② What is r? A: consider all values of r!

Observation ~~VR~~ if $s < t$ then $VR_s(X) \subseteq VR_t(X)$.

$VR_0(X) = X$ ~~$\{x_i\}$~~ $\quad$ for $d > \mathrm{diam}(X)$, $VR_d(X)$ = complete graph / $(|X|-1)$ simplex.

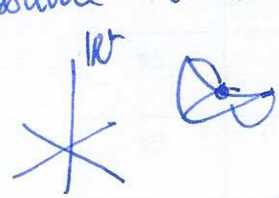
Persistent homology: look for homology groups that last for a long time!

Warmup/ ~~Q~~ sanity check: if we sample points from a known space, can we recover the space? (up to homotopy?).

Niyogi–Smale–Weinberger Thm: (yes).

Start with: compact Riemannian manifold (⇒ smooth metric on manifold), wlog can assume $M \subset \mathbb{R}^n$ for some n.

Recall.

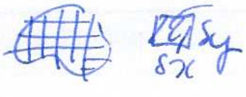


Riemannian metric ⟷ smoothly varying choice of inner product at each point.

- gives: angles, length of tangent vectors.

- integrate lengths of tangent vectors ⇝ get lengths of curves.

- lengths give volumes.



area $\delta x \delta y$

So given $\epsilon > 0$, can work out n s.t. n-points form an ϵ-net with high probability.

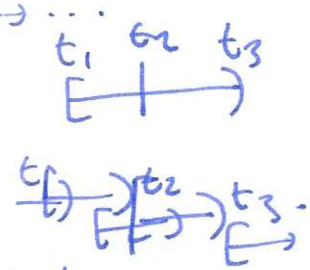
problems: • need to know r (bump around) s.t. tubular nbd $N_r(M)$ is embedded (11)
in $\mathbb{R}^n$. • for $S^1 \subset \mathbb{R}^2$, #points ~ 6000.

Persistent homology setup $X \subseteq \mathbb{R}^n$ (X, d_X) point cloud.

observation: only finitely many t s.t. $VR_t(X)$ changes.

so get sequence $VR_0(X) \hookrightarrow VR_{t_1}(X) \hookrightarrow \cdots$

giv $H_k(VR_0(X)) \xrightarrow{i_1} H_k(VR_{t_1}(X)) \xrightarrow{i_2} \cdots$



persistence

t_1 t_2 t_3.

$\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$. $\Leftarrow$ persistent

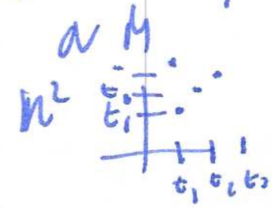
$1 \mapsto 1 \mapsto 1$

$\mathbb{Z} \xrightarrow[0]{} \mathbb{Z} \xrightarrow[0]{} \mathbb{Z}$ $\Leftarrow$ not persistent!

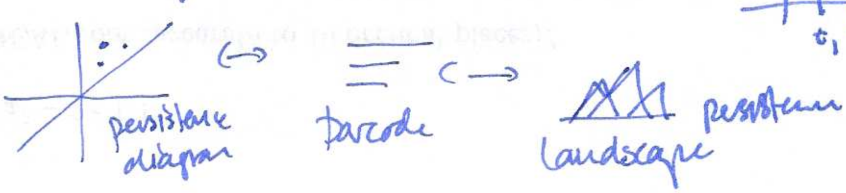
Defn

$\rightsquigarrow$ barcode: collection of intervals of the form $[x,y) \subseteq \mathbb{R}$
$[x,\infty) \subseteq \mathbb{R}$.

$\{$finite metric spaces$\} \to \{$filtered simplicial complexes$\} \to \{$barcodes / persistence diagram$\}$



Examples



Stability

note: Top : top spaces, cts maps

(Ab)Gp : groups, homomorphisms

metric spaces? $\leftarrow$ cts maps.

Lipschitz maps: $d(f(x), f(y)) \leq K d(x,y)$.

(bi)-Lipschitz.

quasi-isometries.

note: there is a metric on subsets $A, B \subseteq (X,d)$.

Def^{n} Let A, B be non-empty subsets of a metric space (X, d) the <u>Hausdorff distance</u> between A and B is

$$d_H(A,B) = \inf_{\epsilon>0} \{ B \subseteq N_\epsilon(A), \ A \subseteq N_\epsilon(B) \}.$$

<u>Equivalently</u> $d_H(A,B) = \max \{ \sup_{a\in A} \inf_{b \in X} d(a,b), \sup_{b\in B} \inf_{a\in A} d(a,b) \}$.

<u>Note</u> $d_H(A,B) = 0 \Rightarrow A = B$.

<u>Examples</u>.

<u>Check metric</u> · $d_H(A,B) = d_H(B,A)$.

· triangle inequality

<u>Example</u>. $\mathbb{Z} \subseteq \mathbb{R}$. $d_H(\mathbb{Z}, \mathbb{R}) = 1/2$.

$Y \subseteq (X,d)$ ϵ-net. $d_H(X,Y) \leq \epsilon$.

Q: when are two metric spaces close?

<u>Problem</u> $(X, d_X), (Y, d_Y)$ not nec. contained in common metric space.

Def^{n} An isometric embedding $f: (X, d_X) \to (Z, d_Z)$ is an injective map s.t. $d_Z(f(x), f(y)) = d_X(x,y)$.

Def^{n} Gromov-Hausdorff distance $(X, d_X), (Y, d_Y)$.

$$d_{GH}(X,Y) = \inf_{\substack{f_1: X \to Z \\ f_2: Y \to Z}} d_H(X,Y)$$

where the infimum is taken over all (Z, f_1, f_2) where $f_1: X \to Z$ and $f_2: Y \to Z$ are isometric embeddings into a metric space Z.

<u>Note</u> X, Y embed in $X \times Y$ with product metric.

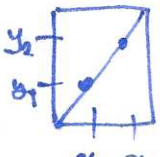
Q: how to compute this?

<u>Fact</u> Let R be a correspondence between X and Y, i.e. $R \subseteq X \times Y$

$(\forall x \in X)\ \exists\ (x,y) \in R$ and $(\forall y \in Y)\ \exists\ (x,y) \in R$.

Then

$$d_{GH}(X,Y) = \inf_{R \subseteq X_1 \times X_2} \frac{1}{2} \sup_{\substack{(x_1,y_1) \in R \\ (x_2,y_2) \in R}} \left| d_X(x_1,x_2) - d_Y(y_1,y_2) \right|$$



↑ i.e. d_{GH} measures maximum distortion of best matching between two metric spaces.

<u>Good.</u> point cloud $X \subseteq \mathbb{R}^n$ ← perturb points ϵ to get X'

$d_{GH}(X, X') \leq \epsilon$

<u>Bad.</u> point cloud $X \subseteq \mathbb{R}^n$, add new point distance K away

then $d_{GH}(X, X') \geq K/2$

So d_{GH}: stable for small perturbations

not stable for random noise far away (outliers)

<u>Recall</u>. $X \rightsquigarrow PH(X) \rightsquigarrow$ barcode.

↑ Q: when are two barcodes close?

<u>Bottleneck distance</u>

given two intervals $[a_1, b_1)$, $[a_2, b_2)$ define

$$d_\infty([a_1,b_1), [a_2,b_2)) = \max\{|a_1 - a_2|, |b_1 - b_2|\}.$$

~~4f~~ for ϕ, define $d_\infty([a,b),\phi) = \frac{|a-b|}{2}$. (14)

Let B_1 and B_2 be barcodes, ~~along~~ ~~$|B_1| \leq |B_2|$~~.
and let $\phi: A_1 \to A_2$ be a bijection from $A_1 \subseteq B_1$ to $A_2 \subseteq B_2$.
extend B_1, B_2 by adding ϕ, and match $B_1 \setminus A_1$ with $\phi \in B_2$
an. $B_2 \setminus A_2$ w $\phi \in B_1$

Defn. Let B_1, B_2 be barcodes.

$$d_B(B_1, B_2) = \inf_{\phi} \sup_{z \in B_1} d_\infty(z, \phi(z)).$$

i.e. this takes the worst discrepancy in ~~the any~~ the best matching

Example. B_1 $[0,1]$ — $[\frac{1}{3},\frac{2}{3}]$.
$[0,\frac{1}{2}]$. — $[\frac{1}{4},\frac{3}{4}]$.
$[0,\frac{1}{4}]$ ~~—~~

example of match

		Δ
$[0,1]$ —	$[\frac{1}{3},\frac{2}{3}]$	$1/3$
$[0,\frac{1}{2})$ —	$[\frac{1}{4},\frac{3}{4}]$.	$1/4$
$[0,\frac{1}{4}]$ —	ϕ.	$1/8$.
$[0,1)$	$[1/4, 3/4]$	$1/4$.
$[0,\frac{1}{2})$	$[1/3, 2/3]$	$1/3$
$[0,\frac{1}{4}]$ —	ϕ.	

$d_B = \frac{1}{3}$.

note adding lots of small bars doesn't change d_B.

do examples of l_p metrics on $\mathbb{R}^2$.

more generally this is the $p = \infty$ (sup norm) version of the Wasserstein metric:

$$d_{W_p}(B_1, B_2) = \left(\inf_{\varphi} \sum_{z \in B_1} \left(d_\infty(z, \varphi(z)) \right)^p \right)^{1/p}$$

↑ note sensitive to lots of small small bars.

Thm [Ch-S-E-H]. Let (X, d_X), (Y, d_Y) be finite metric spaces. Then for all $k \geq 0$.

$$d_B\left(PH_k(VR_\bullet(X)), PH_k(VR_\bullet(Y)) \right) \leq d_{GH}(X, Y).$$

Remark: ~~sim~~ hold for Wasserstein + Čech complex.

— perturbation stability, not outlier stability

Zigzag persistence

take samples X_i from fixed metric space (X, d_X)

$$\begin{tikzcd}
X_1 \arrow[dr, hook] & & X_2 \arrow[dl, hook'] \arrow[dr, hook] & & X_3 \arrow[dl, hook'] & \cdots \\
& X_1 \cup X_2 & & X_2 \cup X_3 & &
\end{tikzcd}$$

$$\begin{tikzcd}
H_k(VR_\varepsilon(X_1)) \arrow[dr, "i_*"] & & H_k(VR_\varepsilon(X_2)) \arrow[dl, "i_*"'] \arrow[dr, "i_*"] & & H_k(VR_\varepsilon(X_3)) \arrow[dl, "i_*"'] \\
& H_k(VR_\varepsilon(X_1 \cup X_2)) & & H_k(VR_\varepsilon(X_1 \cup X_2)) . &
\end{tikzcd}$$

Q: what does consistency mean here?

$$\begin{tikzcd} m_1 \quad H_n(X_1) \arrow[dr, "f_*"] & & H_n(X_2) \quad m_3 \arrow[dl, "g_*"'] \\ & H_n(X_1 \cup X_2) & \\ & m_2 & \end{tikzcd}$$

$f_*(m_1) = m_2 = g_*(m_3)$.

Remark this makes sense for things like

$$\begin{tikzcd} X_1 \arrow[dr] & & X_3 \arrow[dl] \arrow[r] & X_4 \arrow[r] & X_5 \arrow[dr] & & X_7 \arrow[dl] \arrow[r] & \cdots \text{ etc.} \\ & X_2 & & & & X_6 & & \end{tikzcd}$$

Defn- A zigzag submodule N of a zigzag module M of shape S, is a zigmodule of the same shape S s.t. each $N_i \subseteq M_i$ and $f: N_i \to N_j$ is the restriction of $f: M_i \to M_j$.

Lemma Any zigzag module of shape S can be written as a direct sum of indecomposables, this is unique up to permutation.

Defn- An interval zigzag module of shape S is a zigzag module $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} X_k$ ~~with~~ and for $a < b$

$\begin{cases} X_i = \mathbb{F} & \text{if } 1 \le a \le i \le b \le k \\ X_i = 0 & \text{else.} \end{cases}$ and the maps between the $\mathbb{F}$'s are identity

Thm The indecomposable zigzag modules are the interval zigzag modules. (17)

$\rightsquigarrow$ gives something like a persistent barcode.

Computational issues $|X| = n$ $\quad \Delta^n$ has 2^n subsimplices
$\quad d(x)^n$.

solutions: - compute $\underset{n\text{-simple}}{H_0}, \underset{\binom{n}{2} \sim n^2}{H_1}, \underset{n^3}{H_2}$

- take smaller samples.
need some probability theory

Metric measure spaces:

In order to sample a space, we need a measure on the space

Canonical example $X \subseteq \mathbb{R}^n$ finite set of points w/ counting measure.

Problem if X is infinite, can't usually assign measures to all subsets $U \subseteq X$, so measure theory makes choosing special subset of sets called a σ-algebra $\Sigma \subseteq \mathcal{P}(X)$

Properties: $\cdot\ \phi \in \Sigma$
$\cdot$ countable unions
$\cdot$ complements.

Def: A measurable space is a set X and a σ-algebra Σ (X, Σ).

Example Borel σ-algebra: (X, T) topological space is the σ-algebra generated by open sets. (18)

<u>Fact</u> every topological space is a measurable space w/ Borel σ-alg s.t. every open (and closed) subset is measurable.

<u>Eg</u> if X countable, can choose $\Sigma = P(X)$ (i.e. every subset)

$\mathbb{R}^n$, standard top

X simplicial complex, standard top.

<u>~~Defn~~</u> ~~Notation~~ <u>Defn</u> A <u>measure</u> μ on (X, Σ) is a function $\mu : \Sigma \to \mathbb{R}^{\geq 0}$ s.t. $\mu(\phi) = 0$

<u>countable unions</u> work

if $\{X_i\}_{i \in I}$ ~~X_i~~ $\in \Sigma$ w/ $X_i \cap X_j = \phi$ $\forall i, j \in I$ then

countable

$$\mu(\cup X_i) = \Sigma \mu(X_i)$$

<u>Examples</u> · counting measure on a discrete set

· Lebesgue meas on $\mathbb{R}^n$

<u>Defn</u> (X, Σ, μ) is a <u>probability measure</u> if $\mu(X) = 1$

<u>Examples</u> · X = finite set $x_1 \dots x_n$, counting measure

so $\mu(x_i) = \frac{1}{n}$ $1/n$

· X = finite set $x_1 \dots x_n$ $p_i \geq 0$

put wts $p_1 \dots p_n$ $\Sigma p_i = 1$.

- $X = [0,1]$ Lebesgue meas $\mu([a,b]) = b-a$, $a \leq b$.

- $X = [0,1] \times [0,1]$ Lebesgue meas $\mu([a,b] \times [c,d]) = (b-a)(d-c)$.

ex.

<u>Conditional measures</u> Let $A \subseteq (X, \mu)$. w/ $\mu(A) > 0$ then (A, μ_A) is a prob space w if $B \subseteq A$

$$\mu_A(B) = \frac{\mu(B)}{\mu(A)}.$$

<u>probability density function</u>.

$[0,1]$ $(I, \mathcal{A}, \lambda) = ([0,1],$ Lebesgue meas$)$

let $f : [0,1] \to \mathbb{R}$ s.t. $\mu_f(A) = \int_A f \, d\mu$.

<u>pull back/measures</u>. push forw

$f : X \to (Y, \mu)$. $\quad$ $f : (X, \mu) \to Y$.

f measurable. $\quad$ $\mu_f(A) = \mu(f(A))$ $\quad$ $\mu_f(A) = \mu(f^{-1}(A))$.

(mean $f : (X, \Sigma_X) \to (Y, \Sigma_Y)$

s.t. $f^{-1}(\cancel{\text{for}}\ A \in \Sigma_Y) \in \Sigma_X$.

<u>example</u> Gaussian meas on $\mathbb{R}$: $\frac{1}{\sqrt{2\pi}} e^{-x^2}$. $\quad \int_{-\infty}^{\infty} f(x)\,dx = 1$

$\mathbb{R}^n$

Defn A metric measure space is a probability measure is a metric space (X, d_X) (complete, separable) w/ a Borel probability measure μ_X. ~~s.t~~ (i.e. $\mu_X(X) = 1$).

Defn the support of μ_X is a subset $Y \subseteq X$ s.t. for every open nbd U of Y $\mu_X(U) = 1$.

Example · X finite, counting measure
· $([0,1], \lambda)$

Product measures. & (X_1, μ_1) (X_2, μ_2).

then if $A_1 \subseteq X_1$ $A_2 \subseteq X_2$ $\mu_1 \otimes \mu_2 (A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$

~~Countable~~ finite products: $(\prod X_i, \otimes \mu_i)$.

$\mu_i^{\otimes}(A \times$

Warning countable products (simplicity case $(X, \mu)^{\mathbb{N}}$).

on a set $X^{\mathbb{N}} = (x_1, x_2, \ldots)$.

Q: what is $\Sigma_{X^{\mathbb{N}}}$? A: generated by cylinder sets

$X \times X \times \cdots \times A \times \cdots$

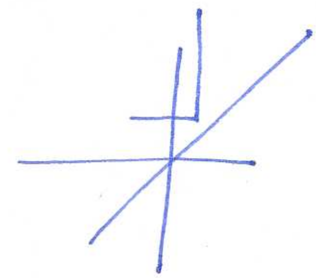
$\uparrow$ $A \subseteq X$ means in ith coordinate.

so $\mu^{\mathbb{N}}(X \times X \cdots \times A \times X \times \cdots X) = \mu(A)$. ㉑

Conclusions

Stability [CS-E-H].

k-triangle Lemma. [ELZ].



f tame function $x < y$ regular points.

multiplicity in upper left quadrant is

$$\#(D(f) \cap Q_x^y) = \beta_x^y$$

$D(f)$: persistence diagram

Q_x^y: quadrant

Defn X top space, $f: X \to \mathbb{R}$.

$a \in \mathbb{R}$ is a homological critical value if $\exists k$ s.t. $\forall \epsilon > 0$

$H_k(f^{-1}(-\infty, a-\epsilon]) \to H_k(f^{-1}(-\infty, a+\epsilon])$ induced by inclusion is not an isomorphism.

$$F_x = H_k(f^{-1}(-\infty, x])$$

for $x < y$ $\quad f_x^y : F_x \to F_y$ map induced by inclusion.

$F_x^y = \mathrm{im}\, f_x^y$

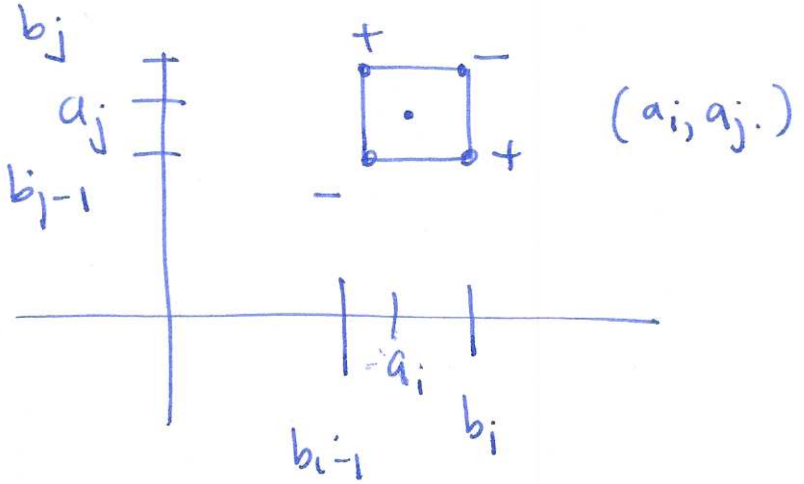
(a_i) critical values

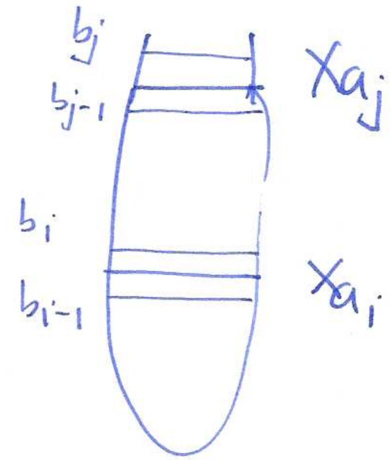
(b_i) interleaved sequence. $b_{i-1} < a_i < b_i$

$b_{-1} = a_0 = -\infty$. $b_{n+1} = a_{n+1} = \infty$.

multiplicity of (a_i, a_j) is $i < j$.

$$\mu_i^j = \beta_{b_{i-1}}^{b_j} - \beta_{b_i}^{b_j} + \beta_{b_i}^{b_{j-1}} - \beta_{b_{i-1}}^{b_{j-1}}.$$





$$\underbrace{\beta_{b_{i-1}}^{b_j} - \beta_{b_{i-1}}^{b_{j-1}}}_{\geq 0} - \Big(\underbrace{\beta_{b_i}^{b_j} - \beta_{b_i}^{b_{j-1}}}_{\geq 0} \Big).$$

# of classes

born in $[b_{i-1}, b_i]$ which die in $[b_{j-1}, b_j]$.