Application $\mathbb{R}^1 \not\cong \mathbb{R}^2$ ← note: there is a bijection from $\mathbb{R}^1$ to $\mathbb{R}^2$.
there is an onto cts map from $\mathbb{R}^1$ to $\mathbb{R}^2$.

Proof suppose $f : \mathbb{R}^1 \to \mathbb{R}^2$ is a homeomorphism. Then this gives a homeomorphism

$f : \mathbb{R}^1 \setminus \{0\} \to \mathbb{R}^2 \setminus f(0)$.

← one connected component. ⨳ □

↑ two connected components

Aim: generalise this argument to higher dimensions (homology) ...

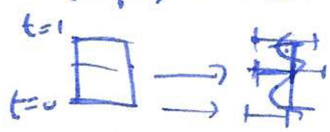
Topological spaces up to homotopy

Defn we say X, Y are homotopy equivalent $X \simeq Y$ if there are maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that $gf \simeq \mathbb{1}_X$ and $gf \simeq \mathbb{1}_Y$.

Special case X is contractible if $X \simeq \{x\}$ point

Example $I = [0,1]$ is contractible. define $F : I \times I \to I$.
$(x,t) \mapsto (1-t)x$

then $f_0(x) = F(x,0) = x = \mathbb{1}_I$.
$f_1(x) = F(x,1) = 0$



Exercises show I^n, $\mathbb{R}^n$, B^n contractible.

(non)-example: $S^0 = \{x\} \sqcup \{y\}$ ← not contractible.

Q: how do we show S^1 not contractible, or any other space for that matter?

variant: retractions. Let $A \subset X$ then X retracts to A is there is a map

$A \underset{i}{\overset{r}{\rightleftarrows}} X$ s.t. $r|_A = \mathbb{1}_A$

we've shown: $[0,1]$ retracts to $[0]$.

Bad example there is a contractible space X which does not retract to any point

Not so bad example comb space  ← retracts to a but not to b.

Q: how does # of path components behave under cts maps?

Propn $f : X \to Y$ cts, then # of path components of $f(X) \leq$ # of path components of X.

Proof if $\gamma : I \to X$ is a path from $\gamma(0)$ to $\gamma(1)$, then $f\gamma : I \to Y$ is a path from $f\gamma(0)$ to $f\gamma(1)$, $f(x)$ to $f(y)$. □