• reflexive: let $f: I \to x \in X$, this is a path from x to x so $x \sim x$.

• symmetric: given $f: I \to X$, a path from $x = f(0)$ to $y = f(1)$

define $\bar{f}: I \to X$ by $\bar{f}(t) = f(1-t)$, the <u>reverse path</u>, is a path from $\bar{f}(0) = y$ to $\bar{f}(1) = x$.

• transitive: let f be a path from x to y, and g be a path from y to z

define $f \cdot g : I \to X$ by $f \cdot g(t) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$

this is a path from x to z, so $x \sim y$ and $y \sim z \Rightarrow x \sim z$

<u>Defn</u> The <u>path components</u> of X are the equivalence classes under this relation.

<u>Remark</u> we have defined our first topological invariant: top spaces $\to \mathbb{N}$.
$X \longmapsto$ # path components of X.

<u>Observation</u> if X homeo Y then # path components of X = # path components of Y.

~~Prop~~ Caution $X = \sqcup X_i$ path components, ~~X_i open and closed~~ ¢.

<u>Propn</u> $S^0 = \{x\} \sqcup \{y\}$ has two path components.

<u>Proof</u> suppose only one, then there is a path $f: I \to S^0$ s.t. $f(0) = x$ and $f(1) = y$
note $\{x\}, \{y\}$ are open in S^0 (in fact an open cover) so $f^{-1}(\{x\})$ and $f^{-1}(\{y\})$ are open in I, and in fact an open cover. Note $0 \in f^{-1}(\{x\})$ and $1 \in f^{-1}(\{y\})$.
$1 \notin$ $0 \notin$.

$f^{-1}(x)$ is a non-empty subset of $[0,1] = I$, contains 0, but not 1, has a supremum = least upper bound s. Recall, for all $\varepsilon > 0$ ~~$t < s$~~, $\exists\, t \in f^{-1}(\{x\})$ s.t. $t \in [s-\varepsilon, s]$.

• suppose $s \in f^{-1}(\{x\})$ ← open, so there is a basis set/open interval $(a,b) \subseteq f^{-1}(\{x\})$ s.t. $s \in (a,b) \subseteq f^{-1}(\{x\})$ but then $\exists\, s+\varepsilon > s$ in $f^{-1}(\{x\})$ # so $s \notin f^{-1}(\{x\})$.

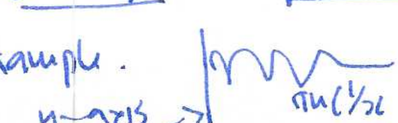
• therefore $s \in f^{-1}(\{y\})$ ← open, so $\exists$ basis set/open interval s.t. $s \in (a,b) \subseteq f^{-1}(\{y\})$, but then $s - \varepsilon$ is bigger than all elements of $f^{-1}(\{x\})$ so s not sup #. $\square$.

<u>Remark</u> we have shown that $I = [0,1]$ is not the disjoint union of two open sets.

<u>Defn</u> X is <u>disconnected</u> if $X = U \sqcup V$, U, V both open.

<u>Connected</u> vs <u>path connected</u> ← not the same!

Example.



← this is <u>connected</u> but not <u>path connected</u>.

<u>Remark</u> no difference for Δ-complexes, cell complexes.