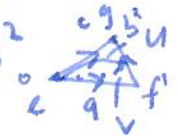
plan: homology $\rightsquigarrow$ persistent homology $\rightsquigarrow$ homotopy $\to \cdots$ 28/3 (1)

recall: X Δ-simplex, $\rightsquigarrow$ chain complex $\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \rightsquigarrow$ homology groups $H_k(X)$

Examples: $X = \{pt\}$ $\cdot$ chain complex $0 \to \overset{1}{\mathbb{Z}} \to \overset{0}{0}$ $\quad H_k(X) = \mathbb{Z}$ $k=0$; 0 $k \geq 1$

$X = S^1$  $0 \to \underset{x}{\mathbb{Z}} \xrightarrow{0} \underset{e}{\mathbb{Z}} \to 0$ $\quad H_k(S^1) = \mathbb{Z}$ $k=0,1$; 0 $k \geq 2$

$[v_0, v_1]$, $[v_0] \sim [v_1]$.

$X = S^2$  $0 \to \underset{u,v}{\mathbb{Z}^2} \to \underset{a,b,c}{\mathbb{Z}^3} \to \underset{e,f,g}{\mathbb{Z}^3} \to 0$

$\partial[u_0,u_1,u_2] = \overset{b}{[u_1,u_2]} - \overset{c}{[u_0,u_2]} + \overset{a}{[u_0,u_1]}$

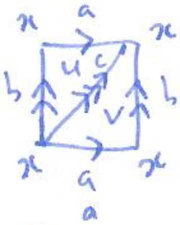
$\partial[v_0,v_1,v_2] =$ "

$\partial \underset{a}{[u_0,u_1]} = \underset{f}{[u_1]} - \underset{e}{[u_0]}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$ $\cancel{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}}$ $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

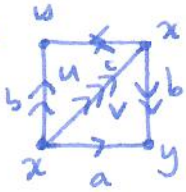
$\sim \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \sim \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$H_k(\mathbb{Z}) = \mathbb{Z}$ $k=0$; 0 $k=1$; $\mathbb{Z}$ $k=2$; 0 $k \geq 0$

$X = T^2$  $0 \to \underset{u,v}{\mathbb{Z}^2} \to \underset{a,b,c}{\mathbb{Z}^3} \xrightarrow[0]{} \mathbb{Z} \to 0$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$H_k(T^2) = \mathbb{Z}$ $k=0$; $= \mathbb{Z}^2$ $k=1$; $= \mathbb{Z}$ $k=2$; 0 $k \geq 3$

$X = \mathbb{RP}^2$  $0 \to \underset{u,v}{\mathbb{Z}^2} \to \underset{a,b,c}{\mathbb{Z}^3} \to \underset{x,y}{\mathbb{Z}^2} \to 0$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$H_k(\mathbb{RP}^2) = \mathbb{Z}$ $k=0$; $\mathbb{Z}_2$ $k=1$; 0 $k=2$; 0 $k \geq 3$

Constructions: $X \vee Y$, CX, $SX \leftarrow$ if X Δ-~~simplex~~ complex, have nice Δ-complex structures.

$X \times Y$ :: 

want: $f: X \to Y \rightsquigarrow f_*: H_k(X) \to H_k(Y)$

Facts: $\cdot$ $f \simeq g \leftarrow$ simplicial map, linear map from each simplex of X to a simplex of Y

$\cdot$ if $f \simeq g$, $f_* = g_*$ on H_k.

Prop^n: If $f: X \to Y$ simplicial, then induce f_*: ~~$X \to Y$~~ $H_k(X) \to H_k(Y)$

let $f_\#: C_k(X) \to C_k(Y)$

$$\sum n_i\sigma_i \mapsto \sum n_i f(\sigma_i) \leftarrow \text{if } \dim(f(\sigma_i)) < k \text{ set } n_i = 0$$

claim

$$\begin{array}{ccccccc} & & & \sigma & \mapsto & \partial\sigma & \\ \cdots & C_{k+1} & \to & C_k(X) & \xrightarrow{\partial} & C_{k-1}(X) & \to \cdots \quad ⊛ \\ & & & \downarrow f_\# & & \downarrow f_\# & \\ \cdots & & \to & C_k(Y) & \xrightarrow{\partial} & C_{k-1}(Y) & \to \cdots \end{array}$$

commutes.

$f_\#\sigma = f\sigma \mapsto \partial f\sigma$

$f_\#\partial\sigma = f_\#\left(\sum(-1)^i[v_0,\dots\hat{v}_i\dots v_n]\right) = \sum(-1)^i f\sigma_i \leftarrow$ i-th face.

Prop^n A chain map is a collection of maps $f_\#: C_k(X) \to C_k(Y)$ s.t. ⊛ commutes induces a homomorphism $f_*: H_k(X) \to H_k(Y)$

defn $[z] \mapsto [f_\# z]$ check well defined

$$\begin{array}{ccccccc} & y \mapsto \partial y & & z & & & \\ \cdots \to & C_{k+1}(X) & \to & C_k(X) & \to & C_{k-1}(X) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ & C_{k+1}(Y) & \to & C_k(Y) & \to & C_{k-1}(Y) & \to \cdots \\ & & & f_\# z & & & \end{array}$$

· check $f_\# z$ cycles

$\partial f_\# z = f_\# \partial z = 0$

· suppose pick z' instead, then $z' = z + \partial y$, $y \in C_{k+1}$

then $f_\#(z+\partial y) = f_\# z + f_\#\partial y = f_\# z + \partial f_\# y$ so $[f_\#(z'+\partial y)] = [f_\# z]$ □.

Example $i: X \subseteq Y$ sub Δ-complex inclusion induces $i_*: H_k(X) \to H_k(Y)$

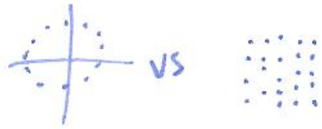
Example $X = \{[v_0],[v_1]\}$ $Y = \{[v_0,v_1]\}$

$k=1$: $0 \to 0$

$k=0$: $\mathbb{Z}^2 \to \mathbb{Z}$

Persistent homology idea: data set $X = \{x_i\} \subseteq \mathbb{R}^d$

let $X_r = \bigcup_i B(x_i, r) \subseteq \mathbb{R}^d$ and let r run from



$r=0$ to $r = \text{diam}(X)$ ← look for homology classes that exist for long ~~periods~~ intervals of r.

X_0 = disjoint pts   ball.

Problem: find $\bigcup B(x_i,r)$ too hard

Solution: use Rips complex. X_r ← connect x_i, x_j if $d(x_i,x_j) < r$

← fill in all possible simplices.

Example  ← no triangle  ← fill in triangle  ← fill in ~~tetrahedron~~ two triangles. 28/3 3

 ← fill in tetrahedron

X_0 ← disjoint collection of points. $X_{\mathrm{diam}(X)}$ ← full $|X_i|$ simplex

↑ exponentially many vertices!

can't compute $H_k(X_r)$ for all k, r

in practice: compute $k = 0, 1, 2$.

• X_r adds a simplex at only finitely many times r_i

so get $X_0 \subset X_{r_1} \subset X_{r_2} \subset \cdots$ increasing union of simplicial complexes.

$$H_k(X_0) \xrightarrow{i_*} H_k(X_{r_1}) \xrightarrow{i_*} H_k(X_{r_2}) \xrightarrow{i_*} \cdots \qquad \cdots \to H_k(X_{r_{100}}) \to \cdots$$

↑ homology class appears here — dies here

↑ get barcode.

long barcodes ⟺ "significant" homology