**Motivation** **Thm** $\mathbb{R}^a \cong \mathbb{R}^b \Rightarrow a = b$. M 27/3 (1)

**Remark:** $I = [0,1]$, $I \times I$ $\square$, there is a bijection $f: I \times I \to I$

$(0.x_1x_2\cdots, 0.y_1y_2\cdots) \mapsto (0.x_1y_1\cdots)$

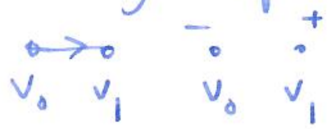
there is a cts map $f: I \twoheadrightarrow I \times I$. (onto)

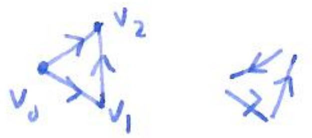
**Thm** Every cts $f: D^n \to D^n$ has a fixed pt.

**Recall** $X = \Delta$-complex $\rightsquigarrow$ chain complex $\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial} 0$

$C_n$ = free abelian group generated by $n$-simplices in $X$.
i.e formal sums of simplices $\sum n_i \sigma_i$

**Def** boundary map $\partial$. $d=0$ $\partial[v_0] = 0$

**d=1**  $\partial[v_0, v_1] = [v_1] - [v_0]$

**d=2**  $\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

**in general** $\partial[v_0, v_1, \dots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \dots, \hat{v}_i, \dots, v_n]$

**d=3**  $\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$

**Propn** $\partial_{n-1}\partial_{n+1} = 0$

**Proof** $\partial[v_0, v_1, \dots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots v_n]$

$\partial\partial[v_0, v_1, \dots, v_n] = \sum_{i=0}^{n} (-1)^i \partial[v_0, \dots \hat{v}_i, \dots, v_n]$

$= \sum_{j<i} (-1)^i (-1)^j [v_0, \dots \hat{v}_j, \dots, \hat{v}_i, \dots v_n] + \sum_{j>i} (-1)^i (-1)^{j-1} [v_0, \dots, \hat{v}_i, \dots \hat{v}_j, v_n]$

$= 0$ $\square$.

**check** $d=2$ $\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

$\partial^2[v_0, v_1, v_2] = [v_2] - [v_1] - [v_2] + [v_0] + [v_1] - [v_0] = 0.$

So $X$ $\Delta$-complex $\rightsquigarrow$ chain complex $\rightsquigarrow$ homology groups $H_k^\Delta(X)$

Fact if $X \cong Y$ then $H_n^\Delta(X) = H^\Delta(Y)$

recall $\cdots \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} \cdots$ has homology group $\ker(\partial_k)/\mathrm{im}(\partial_{k+1})$ 

intuition Defn if $z \in C_k$ has $\partial z = 0$ then we say $z$ is a cycle
if $z \in C_k$ has $z = \partial y$ for $y \in C_{k+1}$ we say $z$ is a boundary

so $H_k(X) = $ "cycles" / "boundaries" ← things with no boundary

Examples.  ← both cycles, but one is a boundary

Examples. ① $X = \{pt\}$ chain complex $\cdots \to 0 \to C_0 \to 0$
$\cdots \to 0 \to \mathbb{Z} \to 0$

$H_0(X) = \mathbb{Z}$
$H_k(X) = 0 \quad k \geq 1$

② $X = $ interval $[0,1] = [v_0, v_1]$. chain complex $\cdots \to 0 \to C_1 \to C_0 \to 0$

$0 \to \mathbb{Z} \xrightarrow[{[1 -1]}]{} \mathbb{Z}^2 \to 0$ $\underset{\text{basis}}{\overset{\text{change}}{\rightsquigarrow}}$ $0 \to \mathbb{Z} \xrightarrow[{(1\ 0)}]{} \mathbb{Z}^2 \to 0$ $\quad 0 \to \mathbb{Z} \to \mathbb{Z}^2 \to 0$
$[v_0, v_1] \overset{\partial}{\mapsto} [v_1] - [v_0]$
$1 \longmapsto 1 - 1$
$[1 -1]$.

↑ injective
$\Rightarrow \ker(\partial) = 0$
$\Rightarrow H_1(X) = 0$

$H_0(X) = \mathbb{Z}^2/\mathrm{im}(\partial) = \mathbb{Z}$

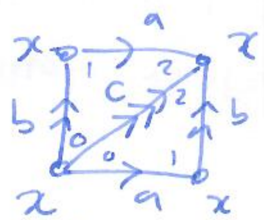
③ $S^1 = [v_0, v_1]/\sim \quad [v_0] \sim [v_1]$

 $\partial [v_0, v_1] = [v_1] - [v_0]$

$\cdots 0 \to C_1 \to C_0 \to 0$
$\mathbb{Z} \xrightarrow[0]{\partial} \mathbb{Z}$

so $H_k(X) = \mathbb{Z} \quad k = 0, 1$
$H_k(X) = 0 \quad k \geq 2$

④ $T^2 = S^1 \times S^1 \quad \cdots \to 0 \to C_2 \to C_1 \to C_0 \to 0$
$\mathbb{Z}^2 \quad \mathbb{Z}^3 \quad \mathbb{Z}$




$\partial [v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$
$a - c + b$

$\partial [w_0, w_1, w_2] = [w_1, w_2] - [w_0, w_2] + [w_0, w_1]$
$b - c + a$

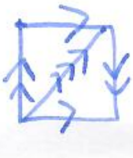
$\partial [v_0, v_1] = [v_1] - [v_0]$
$a \quad\quad x - x = 0$

$0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \xrightarrow[0]{} \mathbb{Z} \to 0$

$H_0(T^2) = \mathbb{Z}$
$H_1(T^2) = \mathbb{Z}^2$
$H_2(T^2) = \mathbb{Z}$

$\begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

⑤ $K^2$ 

Facts $X \simeq Y \Rightarrow H_k(X) \cong H_k(Y)$.
$f: X \to Y$ simplicial induces
$f_*: H_k(X) \to H_k(Y)$

3-d. solid ball $\partial B^3 = S^2$ $\quad \partial\partial B^3 = \partial S^2 = \phi$.

<u>chain complex</u> $\cdots \to C_3 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \to 0$

definin boundary map on each simplex. $\partial[v_0] = 0$

$\partial[v_0, v_1] = [v_1] - [v_0]$

$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

in general: $\partial[v_0, v_1, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$.

$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + (v_0, v_1, v_3] - [v_0, v_1, v_2]$

extend linearly over $C_n$, gives map $\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$

<u>Prop<sup></sup></u>: $\partial_n \partial_{n+1} = 0$

<u>Proof</u> $\partial[v_0, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots v_n]$.

$$\partial\partial[v_0, \ldots v_n] = \sum_{i=0}^{n} (-1)^i \partial[v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

$$= \sum_{0 \le i < j} (-1)^i (-1)^{j-1} [v_0, \ldots, \hat{v}_i, \ldots \hat{v}_j \ldots].$$

$$\sum_{j < i} (-1)^i (-1)^j [v_0, \ldots, \hat{v}_j, \ldots \hat{v}_i \ldots]$$

cancel in pairs! $= 0$

So $\Delta$-complex $\leadsto$ chain complex $\to$ the homology groups $H_k^{\Delta}(X)$.

<u>Th<sup></sup></u> If $X \simeq Y$ then $H_k(X) = H_k(Y)$ (algebraic invariant)

<u>Intuition for homology</u> $C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$ $\quad \partial z = 0$ $\quad$ cells

$z$ $\quad \partial = 0$ cycles / boundaries. $\mathrm{im}(\partial_{n+1})$

$0 \qquad \neq 0$