Calculation w/ free abelian groups = linear algebra over $\mathbb{Z}$ instead of $\mathbb{R}$ ①

recall $\alpha : \mathbb{Z}^n \to \mathbb{Z}^m$ homomorphism $\Rightarrow$ α is a linear map over $\mathbb{Z}$:

$$\alpha(\underline{x} + \underline{y}) = \alpha(\underline{x}) + \alpha(\underline{y})$$

$$\alpha(a\underline{x}) = \underbrace{\alpha(x) + \alpha(x) + \cdots \ \alpha(x)}_{a \text{ times.}} = a\alpha(x).$$

$\uparrow$ $a \in \mathbb{Z}$!

recall Choose bases $e_1, \ldots, e_n$ for $\mathbb{Z}^n$, $f_1, \ldots, f_m$ for $\mathbb{Z}^m$ can write α as a matrix A s.t. $\alpha(x) = Ax$ $\quad x \in \mathbb{Z}^n$ so $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$

$$\text{so } \alpha(x) = \alpha(\text{——}) = \alpha(x_1e_1) + \alpha(x_2e_2) + \cdots + \alpha(x_ne_n)$$
$$= x_1\alpha(e_1) + x_2\alpha(e_2) + \cdots + x_n\alpha(e_n)$$

$$\text{so } A = \begin{bmatrix} | & | & & | \\ \alpha(e_1) & \alpha(e_2) & \cdots & \alpha(e_n) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

columns of A are images of basis vectors under α.

$\alpha(e_1) = a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m$ $\quad$ $\alpha(e_2) = a_{12}f_1 + a_{22}f_2 + \cdots + a_{m2}f_2$

$$\text{so } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{bmatrix}$$

$$\alpha : \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^m$$

can change basis here $\quad$ can change basis here.

warning $\alpha : V \to W$ ~ 2 bases } totally different!
$\alpha : V \to V$ ~ only one basis!

②

$\alpha: \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^m$

① $\uparrow$ basis $e_1, \dots, e_n$ $\quad \uparrow$ basis $f_1, \dots, f_m$

$A = \begin{bmatrix} | & & | \\ \alpha(e_1) & \dots & \alpha(e_n) \\ | & & | \end{bmatrix}$

when $\alpha(e_1) = a_{11} f_1 + a_{21} f_2 + \dots + a_{m1} f_m$ etc.

$\uparrow$

change ~~replace~~ to $e_1, e_2 + e_1, e_3, \dots, e_n$

Q: how does this change A?

$$\begin{bmatrix} | & | & | & & | \\ \alpha(e_1) & \alpha(e_1+e_2) & \alpha(e_3) & \dots & \alpha(e_n) \\ | & | & | & & | \end{bmatrix}$$

$\alpha(e_1) + \alpha(e_2)$

this is a <u>column operation</u>

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & a+c \end{bmatrix}$$

② $\alpha: \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^m$ basis $(f_1, \dots, f_m) \xrightarrow[\text{to}]{\text{change}} (f_1, f_2 + f_1, f_3, \dots, f_m)$ $\quad f_2' = f_2 - f_1$

$A = \begin{bmatrix} a_{11} a_{12} \dots \\ \\ \end{bmatrix}$ wh $\alpha(e_1) = a_{11} f_1 + a_{21} f_2 + \dots + a_{m1} f_m = a_{11} f_1' \circledast f_2' + a_{31} f_3 + \dots + a_{m1} f_m$

$\circledast = a_{21} - a_{11}$ ~~$a_{11} - a_{21}$~~

<u>row operation</u>

$$A = \begin{bmatrix} - r_1 - \\ - r_2 - \\ \vdots \\ r_m \end{bmatrix} \sim \begin{bmatrix} - r_1 - \\ - r_2 - r_1 - \\ r_3 \\ \vdots \\ r_m \end{bmatrix}$$

given by $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c-a & d-b \end{bmatrix}$

<u>summary</u> $\alpha: \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^m$

$\uparrow$ change basis $\nearrow$ get ~~SA~~ $LAU \Leftarrow$ upper triangular w/ 1s on diag

$\uparrow$ lower triangular w/ 1s on diagonal.

<u>recall</u> Gaussian elimination:

can make A upper triangular using only row ops

$$\begin{bmatrix} * & \\ 0 & * \end{bmatrix}$$

$\Leftarrow$ now do column ops! get: $\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$.

Examples $\quad \alpha : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \qquad A = \begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 \\ -2 & -2 \end{bmatrix}$

$$\begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\ker(\alpha) \cong \mathbb{Z}$

$\mathrm{im}(\alpha) \cong 2\mathbb{Z} \leq \mathbb{Z} \oplus \mathbb{Z}$

so $\mathbb{Z}^2 / \mathrm{im}(\alpha) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$

$$\begin{bmatrix} 3 & 4 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

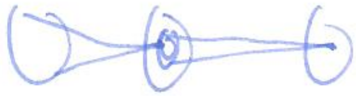
$\ker(\alpha) = 0$

$\mathrm{im}(\alpha) = \mathbb{Z} \oplus 2\mathbb{Z} \leq \mathbb{Z} \oplus \mathbb{Z}$

$\mathbb{Z}^2 / \mathrm{im}(\alpha) = \mathbb{Z}/2\mathbb{Z}$.

Defn A chain complex is a sequence of free abelian groups and maps between them $\cdots \to C_{n+1} \xrightarrow{\alpha_{n+1}} C_n \xrightarrow{\alpha_n} C_{n-1} \to \cdots$ s.t. composition of any two maps is zero, i.e. $\alpha_n \alpha_{n+1} = 0 \ \forall n$.

Observation $\xrightarrow{\alpha_{n+1}} C_n \xrightarrow{\alpha_n}$ $\qquad \alpha_n \alpha_{n+1} = 0 \Rightarrow \mathrm{image}(\alpha_{n+1}) \leq \ker(\alpha_n)$



but don't need to be equal!

$\mathrm{image}(\alpha_{n+1}) \lneqq \ker(\alpha_n)$

so there is an abelian group $\ker(\alpha_n) / \mathrm{Im}(\alpha_{n+1})$

Defn the n-th homology gp of the chain complex is $H_n = \ker(\alpha_n) / \mathrm{im}(\alpha_{n+1})$

Observation we can can compute H_n!

$$C_{n+1} \xrightarrow{\alpha_{n+1}} C_n \xrightarrow{\alpha_n} C_{n-1}$$

diagonalise $\qquad$ diagonalise

$$\left[\begin{array}{c|c} D_{n+1} & 0 \\ \hline 0 & 0 \end{array}\right] \qquad \left[\begin{array}{c|c} D_n & 0 \\ \hline 0 & \underbrace{0}_{\text{size of kernel} \cong \mathbb{Z}^a} \end{array}\right] \qquad H_n \cong \mathbb{Z}^a / D_{n+1}(\mathbb{Z}^b).$$