Intro to groups **Def^n** A **group** is a set G, with a group **operation** $G \times G \to G$, $(g,h) \mapsto gh$ ⓪

with the following properties: **identity** there is a special element $1 \in G$ s.t. $1g = g1 = g$

inverses for every $g \in G$ there is an element $g^{-1} \in G$ s.t. $gg^{-1} = 1 = g^{-1}g$

associativity for all $g,h,k \in G$ $(gh)k = g(hk)$.

Examples $(\mathbb{Z},+)$ $(\mathbb{R},+)$ $(\mathbb{R}\setminus\{0\}, \times)$ $(\mathbb{R}^n, +)$ $(M_{m\times n}, +)$ $m \times n$ matrices

$GL(2,\mathbb{R})$ = 2×2 matrices with $\det \neq 0$, matrix mult. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$SL(2,\mathbb{Z})$ $\det = \pm 1$. permutations

$(f:\mathbb{R}\to\mathbb{R}, +)$ $f+g$, $id = 0$ $(f:\mathbb{R}\to\mathbb{R}, \text{composition})$ fg, $id = f:\mathbb{R}\to\mathbb{R}$, $x \mapsto x$ X any top space $(f: X \to X, \text{composition})$ $F_2 = \langle a, b \mid \ \rangle$.

Def^n A homomorphism $\phi: G \to H$ is a map s.t. for all $g,h \in G$

$$\phi(gh) = \phi(g)\phi(h) \quad (\text{or } \phi(g+h) = \phi(g) + \phi(h)).$$

Examples $\phi_d: (\mathbb{Z},+) \xrightarrow{\times d} (\mathbb{Z},+)$, $n \mapsto dn$. $\phi: (\mathbb{R}^n,+) \xrightarrow{A} (\mathbb{R}^n,+)$, $v \mapsto Av$. A any $n\times n$ matrix.

Def^n A subset $H \subseteq G$ is a **subgroup** if H is a group using the multiplication for G, i.e. $1 \in H$, and for any $h \in H$, $h^{-1} \in H$, and for any $h_1, h_2 \in H$ $h_1 h_2 \in H$.

Examples $2\mathbb{Z} \subset \mathbb{Z} \subset \mathbb{R}$, $SL(2,\mathbb{Z}) \subset GL(2,\mathbb{R})$

Prop^n if $\phi: G \to H$ is a homomorphism, then $\phi(G) = \{h \in H \mid \exists g \in G \text{ w/ } \phi(g) = h\}$ is a subgp of H

and $\ker(\phi) = \{g \in G \mid \phi(g) = 1_H\}$ is a subgp of G.

Constructions · **direct product/direct sum**. $G \oplus H = \{(g,h) \mid g \in G, h \in H\}$

with operation $(g_1,h_1) + (g_2,h_2) = (g_1+g_2, h_1+h_2)$.

free products $G * H$ = all words in $g \in G$, $h \in H$ can cancel gg^{-1}, $g^{-1}g$, hh^{-1}, $h^{-1}h$.

quotients $H \le G$ subgp.

Def: the left cosets of H are the subsets gH for $g \in G$

Remark if G abelian $ab = ba$ for all $a, b \in G$ then right Hg for $g \in G$ no difference.

~~Example~~ ~~Remark~~ Example: $2\mathbb{Z} < \mathbb{Z}$, $3\mathbb{Z} < \mathbb{Z}$

Propⁿ: the left cosets partition G so $\exists g_i$ s.t. $G = \bigsqcup_{i \in I} g_i H$.

Proof define an equivalence relation $a \sim b$ if $b \in aH$. check this is equivalence

reflexive: ~~$a \sim a$~~ ~~$a \sim a$~~: H subg so $1 \in H$ so $a = a1 \in aH$ so $a \sim a$ $\forall a$.

symmetric: sps $a \sim b$, i.e. $b \in aH$ then $b = ah$ for some $h \in H$. subgp

so $h^{-1} \in H$ so $bh^{-1} = a$ so $a \in bH$ so $b \sim a$

transitive: sp. $a \sim b$, $b \sim c$, $b = ah_1$, $c = bh_2$ so $c = \cancel{b}\, a\underbrace{h_1 h_2}_{\in H}$ so $c \in aH$ $\Rightarrow a \sim c$ ∎

Q: do the cosets form a group? Would like $(aH)(bH) = abH$.

Doesn't work unless: H is normal in G, i.e. $gH = Hg$ $\forall g \in G$

then $(aH)(bH) = a(Hb)H = abHH = abH$ ✓.

observation if G is abelian all subgps normal, so H/G always defined.

Examples $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$. write $\mathbb{Z}_2$ $\mathbb{Z}_3$ $\mathbb{Z}_n$.

examples of abelian gps $\mathbb{Z}_n$ $\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$ $\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n} = \mathbb{Z}^n$

$\mathbb{Z}^n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ etc..