Applications

Difference equations interest: P_k = amount of money

$$P_{k+1} = (1.05)P_k$$

solution $P_k = (1.05)^k P_0$

Fibonacci numbers $1, 1, 2, 3, 5, 8, \ldots$

$$F_{n+2} = F_{n+1} + F_n$$

set $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ then $u_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} F_{k+1} + F_k \\ F_{k+1} \end{bmatrix}$

so $u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$ $\quad u_{k+1} = Au_k \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

closed form: $u_k = A^k u_0 \quad u_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

recall if $A = SDS^{-1}$ then $A^k = SD^kS^{-1}$

eigenvalues: $(1-\lambda)(-\lambda) - 1 = 0 \quad \lambda^2 - \lambda - 1 = 0$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

(note $\Rightarrow F_k = a\left(\frac{1-\sqrt{5}}{2}\right)^k + b\left(\frac{1+\sqrt{5}}{2}\right)^k$)

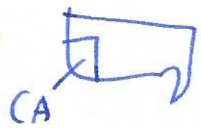
eigenvectors: $\lambda = \frac{1-\sqrt{5}}{2}$ $\quad \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{1}{2} + \frac{\sqrt{5}}{2} \end{bmatrix} \quad \begin{bmatrix} -1 \\ \frac{1}{2} + \frac{\sqrt{5}}{2} \end{bmatrix}$.

$\lambda = \frac{1+\sqrt{5}}{2}$ $\quad \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix} \quad \begin{bmatrix} -1 \\ \frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix}$.

$$D = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 0 \\ 0 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \qquad S = \begin{bmatrix} -1 & -1 \\ \frac{1}{2} + \frac{\sqrt{5}}{2} & \frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix}.$$

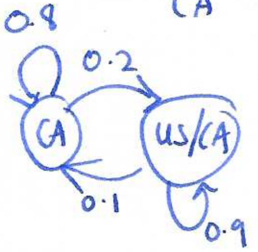
Markov chains

Example



each year $\frac{1}{10}$ of people outside CA move to CA

$\frac{2}{10}$ inside CA move out.



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

x_n = # people in US\CA at time n

y_n = # people in CA.

↑ properties: columns add up to 1

non-negative entries.

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

find eigenvalues: $\det(A - \lambda I) = \lambda^2 - 1.7\lambda + 0.7 = (\lambda - 1)(\lambda - 0.7)$

$$SDS^{-1} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = SD^n S^{-1} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (0.7)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= (x_0 + y_0) \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} + \underbrace{(x_0 - 2y_0)(0.7)^n}_{\to 0 \text{ as } n \to \infty} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}.$$

$$\to \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} (y_0 + x_0).$$

Population example if ~~A~~ $x_n = A^n x_0 = SD^n S^{-1} x_0$

Q: when is population stable? **A**: when $|\lambda_i| \leq 1$ for all i.

unstable if at least one $|\lambda_i| > 1$.

Q: what should e^{At} be?

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

define: $e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots$

$$= e^{SDS^{-1}t} = I + SDS^{-1}t + SD^2S^{-1}\frac{t^2}{2!} + \cdots$$

$$= S\left(I + Dt + D^2\frac{t^2}{2!} + \cdots\right)S^{-1}.$$

$$= S\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & \ddots & e^{\lambda_n t}\end{bmatrix}S^{-1}$$

Recall: not every matrix can be diagonalized.

Jordan normal form

Def: A Jordan block is a matrix of the form $\begin{bmatrix} \lambda & 1 & 0 & 0 & & 0 \\ 0 & \lambda & 1 & 0 & & \\ 0 & 0 & \lambda & 1 & & \\ & & & \ddots & \ddots & 1 \\ & 0 & & & & \lambda \end{bmatrix}$

Thm: Every matrix A has matrices S, J s.t.

$A = SJS^{-1}$, where $J = \left[\begin{array}{c|c|c} J_1 & 0 & 0 \\ \hline 0 & J_2 & 0 \\ \hline 0 & 0 & J_3 \end{array}\right]$ Jordan blocks on the diagonal.