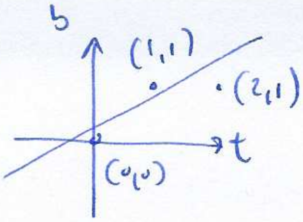
Example



$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \hat{c} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 3 & 2 \\ 0 & 2 & 1 \end{array}\right] \qquad \hat{D} = \frac{1}{2} \qquad \hat{C} = \frac{1}{6}.$$

General case

$$\left[\begin{array}{cc|c} m & \Sigma t_i & \Sigma b_i \\ \Sigma t_i & \Sigma t_i^2 & \Sigma t_i b_i \end{array}\right] \qquad \left[\begin{array}{cc|c} m & \Sigma t_i & \Sigma b_i \\ 0 & \Sigma t_i^2 - \frac{1}{m}(\Sigma t_i)^2 & \Sigma t_i b_i - \frac{1}{m}\Sigma t_i \Sigma b_i \end{array}\right].$$

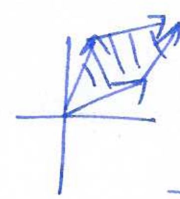
$$\hat{D} = \frac{\Sigma t_i b_i - \frac{1}{m}\Sigma t_i \Sigma b_i}{\Sigma t_i^2 - \frac{1}{m}(\Sigma t_i)^2} \qquad \hat{C} = \frac{1}{m}\Sigma b_i - \Sigma t_i \hat{D}.$$

§ Determinants.

If A is an $n \times n$ matrix, $\det(A)$ is a number. Useful facts:

- A is invertible iff $\det(A) \neq 0$
- $\det(A) = (\pm)$ product of pivots.
- 2×2 case: $\det(A)$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det(A) = ad - bc$
- $\det(A) =$ volume of parallelopiped with sides cols of A.

2x2



3x3

key properties

- $\det(I) = 1$
- row swap changes sign of det, i.e. if $B = A$ with 2 rows swapped $\det(B) = -\det(A)$.
- $\det(A)$ is linear in each row.

notation $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

2x2 case

① $\det(I) = 1$ $\quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

② row exchange $\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad.$

③ linear in each row:

sums: $\begin{vmatrix} a_1 + a_2 & b_1 + b_2 \\ c & d \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ c & d \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ c & d \end{vmatrix}$ check!

scalar mult: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$

<u>warning</u> $\det(\underset{n\times n}{tA}) = t^n \det(A)$!

④ if two rows are equal, then $\det(A)=0$ $\quad \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$.

⑤ adding a multiple of one row to another does not change $\det(A)$.

$$\begin{vmatrix} a+tc & b+td \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + t\begin{vmatrix} c & d \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

⑥ if A has a row of zeros then $\det(A)=0$ $\quad \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0$.

⑦ if A is triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$ = product of diagonal entries. $\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad$.

<u>Proof</u> (general case) $\begin{vmatrix} a_{11} & & * \\ & a_{22} & \\ 0 & & \ddots \; a_{nn} \end{vmatrix} = a_{nn} \det \begin{bmatrix} a_{11} & & \cancel{*} & 0 \\ & a_{22} & & 0 \\ & & \ddots & 0 \\ 0 & & a_{n-1\,n-1} & 0 \\ & & & 1 \end{bmatrix} = a_{11}a_{22}\cdots a_{nn}\,[I]$. □

⑧ if A is singular, then $\det(A)=0$
if A is invertible, then $\det(A) \neq 0$

<u>Proof</u> row reduction gives upper triangular matrix with same determinant (up to sign).

$U = \begin{bmatrix} u_{11} & & \\ & u_{22} & \\ & & u_{nn} \end{bmatrix}$ full set of pivots $\Leftrightarrow$ invertible $\quad \det(A) = u_{11}u_{22}\cdots u_{nn}$

$\leq n-1$ pivots $\Rightarrow$ zero row $\Rightarrow \det(A)=0$ □.

⑨ $\det(AB) = \det(A)\det(B)$

<u>note</u> if $AA^{-1}=I$ then $\det(A)\det(A^{-1}) = 1 \Rightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$

⑩ $\det(A^T) = \det(A)$.

<u>§ Formulas for the determinant</u>

3x3 case: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

4×4 case:
$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Note can expand along any row or column

Example
$$\begin{vmatrix} 0 & -1 & 1 & 2 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 2 & 4 & 1 \end{vmatrix} = 3\begin{vmatrix} 0 & -1 & 2 \\ -1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

Permutations 3×3 case:
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

permutations of $(1\ 2\ 3)$ 123   132   213   231   312   321

Note: there are $n!$ permutations of $(1\ 2\ 3 \ldots n)$

each permutation has a sign $+1$ (even) or -1 (odd)

- even number of swaps: $+1$
- odd number of swaps: -1

Example   1 2 3   notation $\operatorname{sign}(P)$

1 3 2 ← odd

3 1 2 ← even

Remark: can represent permutation by matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} \text{ etc...}$$

Formula for the determinant

$$\det(A) = \sum_{\substack{\text{all permutations} \\ P}} \left(a_{1p(1)} a_{2p(2)} \cdots a_{np(n)} \right) \operatorname{sign}(P).$$

Corollary $A = [(a_{ij})]$ $\det(A)$ is continuous in the a_{ij}.

Applications of the determinant

① methods for finding A^{-1}. (usually quickest $[A \mid I] \xrightarrow{\text{row ops}} [I \mid A^{-1}]$) $O(n^3)$.

Example (2×2) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $A^{-1} = \frac{1}{\det A} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}^T = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$c_{ij} = (ij)$-cofactor $\leftarrow$ remember sign! $(-1)^{i+j}$

in general $(n\times n)$ $A^{-1} = \frac{1}{\det(A)} C^T$ $C = $ matrix of cofactors.

check $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & & \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} \det(A) & & 0 \\ & \ddots & \\ 0 & & \det(A) \end{bmatrix}$

1st row, 1st col: $a_{11}c_{11} + a_{12}c_{21} + a_{13}c_{31} + \cdots + a_{1n}c_{n1} = \det(A)$

1st row, 2nd col: $a_{11}c_{21} + a_{12}c_{22} + \cdots + a_{1n}c_{2n}$

= det of matrix with two rows the same!

$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{11} & \cdots & a_{1n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \leftarrow \det = 0.$

Q: why do we want a formula?

A: shows – entries of A^{-1} are degree n rational functions in entries of A

– $\det(A)$, A^{-1} depend continuously on entries of A.

② solve $Ax = b$ (if A $n\times n$ full rank)