$x = A^{-1}b = \dfrac{1}{\det(A)} C^T b.$

<u>Cramer's rule</u> $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad x_i = \dfrac{\det B_j}{\det A}$ $\quad$ b in j-th-column

$$B_j = \begin{array}{c} \begin{array}{ccccc} & & & \downarrow & \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & a_{1n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & & b_n & a_{nn} \end{bmatrix} \end{array}$$

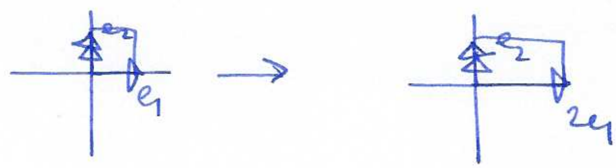
<u>Proof</u> expand along row j. $\square$.

<u>note</u>: solutions vary continuously with coeffs in A

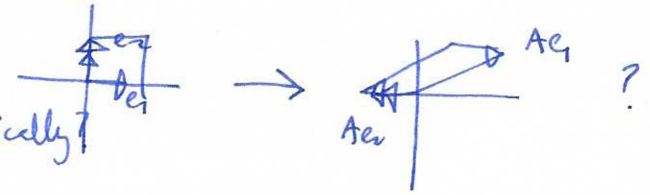
<u>problem</u>: errors become large if $\det(A)$ close to zero.

<u>Eigenvalues and eigenvectors</u>

<u>Example</u> $A : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$



<u>Q</u>: what about $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$?

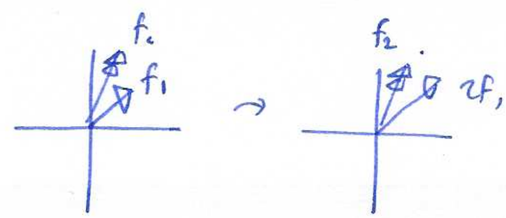


how do we understand this geometrically?

<u>note</u>:

$\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ f_1

$\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ f_2



<u>observation</u> $F = \{f_1, f_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$, so we can write any vector $x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then

$$Ax = c_1 A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 A \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= 2c_1 A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

i.e. $x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_F$ $\quad [Ax]_F = \begin{bmatrix} 2c_1 \\ c_2 \end{bmatrix}$ so $A_F = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

<u>Def</u> An <u>eigenvector</u> v for A is a vector v such that $Av = \lambda v$

λ is called the <u>eigenvalue</u> for v ($v \neq 0$, but $\lambda = 0$ is ok)

Examples $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
eigenvalue 2 1

$\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
eigenvalues 2 1

How to find eigenvectors and eigenvalues

$$Av = \lambda v = \lambda I v \iff Av - \lambda I v = 0 \iff \underbrace{(A - \lambda I)}_{n\times n \text{ matrix}} v = 0$$

v lies in null space / kernel of $A - \lambda I$

recall: $\dim(\text{null}(A - \lambda I)) > 0 \iff \det(A - \lambda I) = 0$

Defn $\det(A - \lambda I)$ is called the characteristic equation for A.

to find eigenvalues, solve $\det(A - \lambda I) = 0$

example $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) = 0$

$\lambda = 2, 1$

$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda) + 2$

$= \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0$

$\lambda = 2, 1.$

note: A $n\times n$, $\det(A - \lambda I)$ is degree n polynomial, so n solutions (counting with multiplicity)

Once you've found the eigenvalues, you can find the eigenvectors.

Example $\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ $\lambda = 1$: solve $(A - I)x = 0$

$$\begin{bmatrix} 3-1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \overset{\text{row}}{\rightsquigarrow} \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

free var t

$2x_1 - t = 0$ $x_1 = t/2$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

$\lambda = 2$: solve $(A - 2I)x = 0$ $\begin{bmatrix} 2-2 & -1 \\ 2 & 0-2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \overset{\text{row}}{\rightsquigarrow} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

free var t

$x_1 - t = 0$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

summary $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ has eigenvalues $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
with eigenvectors 2 1

Observation · if v is an eigenvector, so is any multiple of v $A(cv) = cAv = c\lambda v = \lambda(cv)$

· $\dim(\text{null}(A - \lambda I))$ may be > 1

Example $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ eigenvalues $1, 2$ $\lambda = 1$ has eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\chi_A = (1-\lambda)^2(2-\lambda)$ $\lambda = 2$ has eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Special case upper triangular matrices

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$ $\det(A - \lambda I) = (1-\lambda)(4-\lambda)(6-\lambda)$

so eigenvalues are diagonal elements.

useful facts · $\det(A)$ = product of eigenvalues $\lambda_1 \lambda_2 \cdots \lambda_n$

· $\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ sum of eigenvalues.

warning : row operations do not preserve eigenvalues!

Example $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \leftarrow \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$\chi_A = \lambda^2$ $\chi_A = \lambda(\lambda - 1)$ $\leftarrow$ different!

Examples $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1$

$\lambda = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3 \pm \sqrt{5}}{2}$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$ $\lambda = \pm i$

find eigenvectors: $\begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix} \rightsquigarrow \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix}$ $v = t\begin{bmatrix} 1 \\ -i \end{bmatrix}$ check $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix} = i\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Diagonalization

Example $\begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}$ has eigenvalues 2 -1

with eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

$$\begin{tikzcd} \mathbb{R}^2_E \arrow[r, "A_E"', "\begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}"] & \mathbb{R}^2_E \\ \mathbb{R}^2_F \arrow[u, "S"] \arrow[r, "A_F"'] & \mathbb{R}^2_F \arrow[u, "S"'] \end{tikzcd}$$

$E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ $\quad F = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

Q: what is matrix for A wrt basis F?

<u>A:</u> $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so $A_F = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$A \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Let S be the change of basis matrix from F to E, so $S = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

i.e. S is the <u>matrix of eigenvectors</u> (if there are enough eigenvectors!)

then $A_F = S^{-1} A_E S$ check: $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \leftarrow$ diagonal matrix of eigenvalues!

<u>In general</u> suppose A has n linearly independent eigenvectors $v_1, \dots, v_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Let $S = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$

then $\boxed{S^{-1} A S = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}$ equivalently $A = S \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} S^{-1}$

<u>Remarks</u> 1) if all eigenvalues $\lambda_1, \dots, \lambda_n$ are distinct, then their eigenvectors $v_1, \dots, v_n$ are independent

2) S is not unique.

3) not all matrices have n independent eigenvectors.

problem: repeated eigenvalues.

<u>Example</u> $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \lambda_1 = \lambda_2 = 0 \quad \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0.$

but $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0$, so solutions are $t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

only 1 eigenvector

<u>Def</u> the algebraic multiplicity of an eigenvalue is the power of $(\lambda-\lambda_i)$ in $\det(A-\lambda I)=0$

the geometric multiplicity is $\dim(\text{null}(A-\lambda_i I))$

note: geometric multiplicity $\leq$ algebraic multiplicity

$<$ $\Rightarrow$ not diagonalizable.

Example $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ not diagonalizable. $D=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ but $SDS^{-1}=0$

$\Rightarrow A=0$ ⨯⨯.

Example $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ not diagonalizable $\lambda_1=\lambda_2=1$ $v_1=\begin{bmatrix}1\\0\end{bmatrix}$

$D=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, but then $SDS^{-1}=I\neq A$

Lemma Eigenvectors corresponding to distinct eigenvalues are independent

Proof (2x2 case) Let v_1 have eigenvalue λ_1 $Av_1=\lambda_1 v_1$

v_2 λ_2 $Av_2=\lambda_2 v_2$

suppose ① $c_1v_1+c_2v_2=0$, then $A(c_1v_1+c_2v_2)=0$

$$=c_1Av_1+c_2Av_2$$

$$=\lambda_1c_1v+\lambda_2c_2v_2=0 \quad ②$$

$\lambda_1① - ②$: $\lambda_1c_2-\lambda_2c_2v_2=\underbrace{(\lambda_1-\lambda_2)}_{\neq 0}\underbrace{c_2v_2}_{\neq 0}=0 \Rightarrow c_2=0$

$② - \lambda_2①$: $c_1(\lambda_1-\lambda_2)v_1=0 \Rightarrow c_1=0 \Rightarrow$ linearly indep. □.

examples ① projection matrix: $A=\begin{bmatrix}1/2&1/2\\1/2&1/2\end{bmatrix}$ eigenvalues $(\frac{1}{2}-\lambda)^2-\frac{1}{4}=0$

eigenvalues: $\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}$ $\lambda^2-\lambda=0$

$\lambda(\lambda-1)=0$ $\lambda=0,1$

② rotation: $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ $\det(A-\lambda I)=(-\lambda)^2+1=0$ $\lambda=\pm i$

eigenvectors $\begin{bmatrix}1\\-i\end{bmatrix},\begin{bmatrix}1\\i\end{bmatrix}$ check $S^{-1}AS=D$!