Thm $Q: V \to V$ preserves lengths, i.e. $\|Qx\| = \|x\|$ for any x.

Proof $\|Qx\|^2 = (Qx)^T Qx = x^T Q^T Q x = x^T I x = x^T x = \|x\|^2$ $\square$.

- How to write a vector as the sum of the q_i

$$v = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n$$

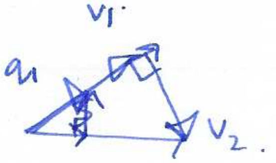
$$q_i \cdot v = q_i^T v = c_1 \underbrace{q_i \cdot q_1}_{0} + c_2 \underbrace{q_2 \cdot q_i}_{0} + \cdots + c_i \underbrace{q_i \cdot q_i}_{1.} + \cdots + c_n \underbrace{q_i \cdot q_n}_{0}$$

so $c_i = q_i \cdot v = q_i^T v$

so $v = (v \cdot q_1) q_1 + (v \cdot q_2) q_2 + \cdots + (v \cdot q_n) q_n$.

Gram-Schmidt: take any basis $\{v_1, v_2, \ldots, v_n\}$ and produce an orthonormal basis $\{q_1, q_2, \ldots, q_n\}$.

① set $q_1 = \frac{v_1}{\|v_1\|}$



problem: v_2 not $\overset{\text{nec}}{\wedge}$orthogonal to q_1. solution: subtract off component of v_2 in direction of ~~v_2~~ q_1. projection of ~~q_1~~ v_2 to q_1 is $\frac{q_1 \cdot v_2}{q_1 \cdot q_1} q_1 = (q_1 \cdot v_2) q_1$.

② set $q_2 = \frac{v_2 - (q_1 \cdot v_2) q_1}{\| v_2 - (q_1 \cdot v_2) q_1 \|}$ now repeat for v_3, subtracting components in directions of q_1, q_2

③ set $q_3 = \frac{v_3 - \text{components in directions of } q_1, q_2}{\| \quad \text{length} \quad \|}$.

$$q_3 = \frac{v_3 - (v_3 \cdot q_1) q_1 - (v_3 \cdot q_2) q_2}{\| v_3 - (v_3 \cdot q_1) q_1 - (v_3 \cdot q_2) q_2 \|}$$

etc.

Example $\left\{ \underset{v_1}{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \right\}$ $\quad q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$q_2' = \overset{v_2}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} - (v_2 . q_1) q_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad q_3' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - (v_3 . q_1) q_1 - (v_3 . q_2) q_2 .$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

check!

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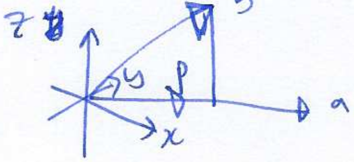
§ Projections



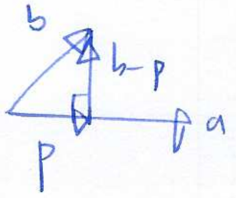
$p =$ projection of b onto (line through) a

recall geometric defn of dot product

$$a \cdot b = a^T b = \|a\| \, \|b\| \cos\theta$$

Key fact: $p = \dfrac{a . b}{a . a} a$

Proof $p = \lambda a$ for some $\lambda \in \mathbb{R}$



$$(b-p) . a = 0$$
$$(b - \lambda a) . a = 0$$
$$a.b - \lambda a.a = 0 \quad \lambda = \frac{a.b}{a.a} \quad \text{so} \quad p = \frac{a.b}{a.a} a \quad \square.$$

Corollary Schwarz inequality

$|a . b| \leq \|a\| \, \|b\|$ with equality iff $a = \pm\lambda b$.

Example project $b = [1 \; 2 \; 3]^T$ onto $a = [3 \; 2 \; 1]^T$

$$p = \frac{a.b}{a.a} a = \frac{3+4+3}{9+2+1} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \frac{10}{12} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \frac{5}{6} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Observation projection is a linear map!

$$\frac{(x+y).a}{a.a} a = \frac{x.a}{a.a} a + \frac{y.a}{a.a} a \qquad \frac{kx.a}{a.a} a = k\left(\frac{x.a}{a.a}\right) a.$$