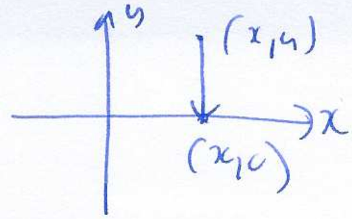
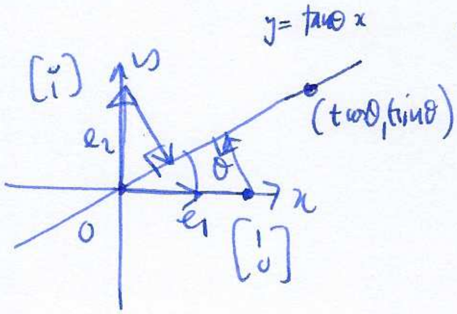
<u>Projections</u> <u>Example</u> $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\0\end{bmatrix}$



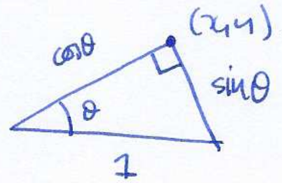
not invertible!

<u>Example</u> projection on to line at angle θ



sufficies to find images of basis vectors $e_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $e_2=\begin{bmatrix}0\\1\end{bmatrix}$

$\begin{bmatrix}1\\0\end{bmatrix}$

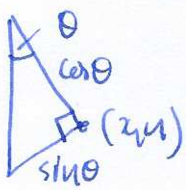


$d((0,0),(t\cos\theta, t\sin\theta)) = t.$

so $t=\cos\theta$

$\Rightarrow (x,y) = (\cos^2\theta, \cos\theta\sin\theta)$

$\begin{bmatrix}0\\1\end{bmatrix}$



$t=\sin\theta$ so $(x,y) = (\sin\theta\cos\theta, \sin^2\theta)$

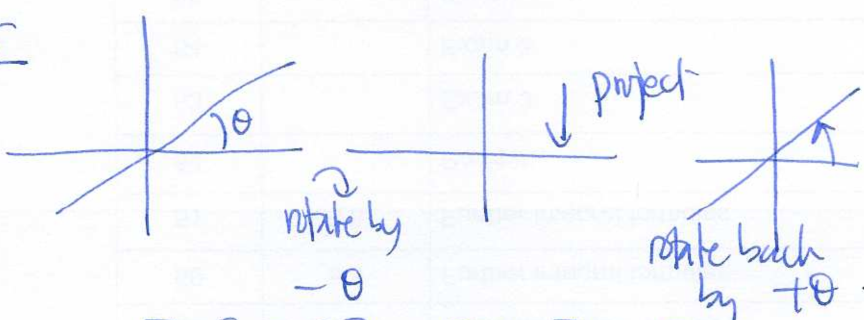
so $A = \begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}$

<u>observation</u> $P^2 = P$

<u>check</u>: $\begin{bmatrix}c^2 & sc\\ sc & s^2\end{bmatrix}^2 = \begin{bmatrix}c^4+c^2s^2 & sc^3+s^3c\\ sc^3+s^3c & s^2c^2+s^4\end{bmatrix}$

$= \begin{bmatrix}c^2(c^2+s^2) & sc(c^2+s^2)\\ sc(c^2+s^2) & s^2(c^2+s^2)\end{bmatrix}$

<u>Better</u>

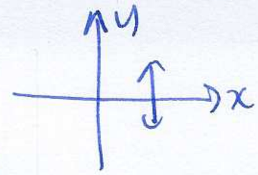


$R_{-\theta} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ $\quad \underset{P.}{\begin{bmatrix}1&0\\0&0\end{bmatrix}}$ $\quad \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ R_θ

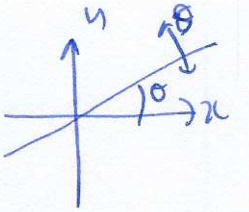
$R_\theta P R_{-\theta} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$

$\begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}$

Reflections



$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -x \\ y \end{bmatrix}$ $\quad H = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ note $H^2 = I$.



$$H_\theta = R_\theta H R_{-\theta} = \begin{bmatrix} 2c^2-1 & 2cs \\ 2cs & 2s^2-1 \end{bmatrix}$$ check: $H_\theta^2 = I$

Q: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ how do we describe action of A on $\mathbb{R}^2$?

Change of basis $\quad V$ vector space basis $B_1 = \{e_1, e_2, \dots e_n\}$

$B_2 = \{v_1, v_2, \dots, v_n\}$.

observation we can write basis B_2 in terms of B_1:

$v_1 = a_{11} e_1 + a_{21} e_2 + \dots + a_{n1} e_n$ $\quad$ in column vector notation:

$v_2 = a_{12} e_1 + a_{22} e_2 + \dots + a_{n2} e_n$ $\quad v_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}_{B_1}, \quad v_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, \quad \dots, \quad v_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix}$

$\vdots$

$v_n = a_{1n} e_1 + a_{2n} e_2 + \dots + a_{nn} e_n$

Let $v \in V$ be a vector. Can write it in basis B_2: $v = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$

but can then write it in basis B_1: $\quad = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{B_2}$

$$v = x_1 (a_{11} e_1 + a_{21} e_2 + \dots + a_{n1} e_n) + x_2 (a_{12} e_1 + a_{22} e_2 + \dots + a_{n2} e_n) + \dots + x_n (a_{1n} e_1 + \dots + a_{nn} e_n)$$

$$v = (a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n) e_1 + (a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n) e_2 + \dots + (a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n) e_n.$$

so $[v]_{B_1} = A [v]_{B_2}$ where $A = \begin{bmatrix} a_{11} & a_{12} & \cdot & a_{1n} \\ a_{21} & & & \\ a & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{B_2}$ $\quad A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$

so columns of A are basis vectors v_i in terms of $B_1 = \{e_1, \dots, e_n\}$

Thm Let $\underset{\{e_1 \dots e_n\}}{B_1}, \underset{\{v_1 \dots v_n\}}{B_2}$ be bases for V, and let A be the matrix ~~of~~ whose columns are $[v_i]_{B_1}$ then $[v]_{B_1} = A [v]_{B_2}$.

Q: how do we change from B_1 to B_2? A: A^{-1}.

Thm: If A is a change of basis matrix, then A is invertible, i.e. A^{-1} exists.

Proof: the map $A: V \to V$, $x \mapsto Ax$ is onto, as every vector $v \in V$ can be written as a sum of basis elements. Furthermore, A is one-to-one, as basis vectors are linearly independent so if $Ax = 0 \Rightarrow x = 0$. Now suppose $Ax = Ay$ then $A(x-y) = 0 \Rightarrow x - y = 0 \Rightarrow x = y$ so one-to-one. □.

Change of basis for linear maps

$T: V \longrightarrow W$.

Let T be given by matrix A wrt B_{V_1}, B_{V_2}.

$B_{V_1} = \{e_1, \dots, e_n\}$. $B_{W_1} = \{f_1, \dots, f_m\}$

$B_{V_2} = \{v_1, \dots, v_n\}$ $B_{W_2} = \{w_1, \dots, w_m\}$.

Q: what is matrix for T wrt B_{V_2}, B_{W_2}?

$$\begin{tikzcd}
Cx \arrow[rrr, mapsto, "T \nearrow A_{B_{V_1}, B_{V_2}}"] & & & ACx \arrow[ddd] \\
& V_{B_{V_1}} \arrow[r] \arrow[d, leftrightarrow] & W_{B_{W_1}} \arrow[d, leftrightarrow, "D"] & \\
& V_{B_{V_2}} \arrow[r, "T"] & W_{B_{W_2}} & \\
x \arrow[uuu, "C"] & D^{-1}AC & & D^{-1}ACx
\end{tikzcd}$$

Let C be change of basis matrix for V

$C = [v_1\ v_2 \cdots v_n]$ ← in basis B_{V_1}

Let D be change of basis matrix of W

$D = [w_1\ w_2 \cdots w_m]$ ← in basis B_{W_1}

$T_{B_{V_2}, B_{V_2}}$ is given by matrix $D^{-1}AC$

Special case $T: V \to V$ linear map from V to itself. $B_{V_1} = \{e_1, \dots, e_n\}$, $B_{V_2} = \{v_1, \dots, v_n\}$

only two bases! let $C = [v_1 \dots v_n]$ ← in basis B_{V_1}. be change of basis matrix

$$\begin{tikzcd}
Cx \arrow[rrr, mapsto] & & & ACx \arrow[ddd] \\
& V \arrow[r, "T"', "A"] & V & \\
& V \arrow[u, "C"] \arrow[r, "T"] & V \arrow[u, "C"'] & \\
x \arrow[uuu] & & & C^{-1}ACx
\end{tikzcd}$$

so if T_{B_1} has matrix A

T_{B_2} has matrix $C^{-1}AC$.

Q: $V, W \subseteq \mathbb{R}^n$ have basis $B_V = \{v_1, \dots v_k\}$ does $V = W$?
$B_W = \{w_1, \dots, w_k\}$.
pick basis $E = \{e_1, e_2 \dots e_n\}$.

A: if $V = W$, every vector in W can be written as a linear combination of the v_i

so $w_1 = x_1 v_1 + x_2 v_2 + \dots + x_k v_k$, i.e. $Cx = w_1$

when $C = [v_1 \dots v_k]^{n \times k}$ ← in basis $E = \{e_1, e_2 \dots e_n\}$.

so want to solve: $Cx = w_1$, $Cx = w_2$, $\vdots$, $Cx = w_k$.

let $D = [w_1, \dots w_k]^{n \times k}$ ← in basis $E = \{e_1, e_2, \dots, e_n\}$

So row reduce $[C | w_1 w_2 \dots w_k] = [C | D]$ ← $n \times 2k$.
$\left[\begin{array}{c|c} \text{blocks} & \\ 0 & \end{array}\right]$

consistent ⇒ $V = W$ inconsistent ⇒ $V \neq W$.

Special bases: orthogonal | orthonormal | Gram-Schmidt.

$\mathbb{R}^n$, standard inner product V vector space $\langle \cdot, \cdot \rangle$ inner product.

Def^n A basis $\{v_1, \dots, v_n\}$ for V is orthogonal if $v_i \cdot v_j = 0$ for all $i \neq j$.
Def^n A basis $\{v_1, \dots, v_n\}$ for V is orthonormal if it is orthogonal and $\|v_i\| = 1$ for all i.

Example $\mathbb{R}^n$, $\{e_1, e_2, \dots, e_n\}$ $\mathbb{R}^2$, $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ $\mathbb{R}^2$, $\left\{\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\right\}$.

Def^n We say a square matrix $Q_{n \times n}$ is orthogonal if the columns of Q are an orthonormal basis for $\mathbb{R}^n$

Recall Q is a change of basis matrix.

Useful fact $Q^T Q = I$ Pf: $\begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \;1 \end{bmatrix} = I_n$

Examples $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $Q =$ any permutation matrix.