$T : V \to W$ column vector notation

choose a basis $\overset{B_V}{v_1, \dots, v_n}$ for V $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{B_V} = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$.

B_W $w_1, \dots, w_m$ for W

then $T(v_i) \in W$, so
$$T(v_1) = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}.$$
$$T(v_2) = a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m.$$
$$\vdots$$
$$T(v_n) = a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m$$

let $v \in V$ be a vector. Then $v = x_1 v_1 + x_2 v_2 + \dots + x_n v_n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

$$\begin{aligned} T(v) &= T(x_1 v_1 + \dots + x_n v_n) \\ &= T(x_1 v_1) + \dots + T(x_n v_n) \\ &= x_1 T(v_1) + \dots + x_n T(v_n). \\ &= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \end{aligned}$$

so let A be the matrix $\begin{bmatrix} a_{ij} \end{bmatrix}$ or $(A)_{ij} = a_{ij}$.

then $T(v) = Ax$ or
$$T : V \to W$$
$$v \mapsto T(v)$$
$$\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{B_V}}_{n \times 1} \mapsto \underbrace{\underbrace{[A]}_{m \times n} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{n \times 1}}_{m \times 1}. \leftarrow \text{in basis } B_W.$$

<u>claim</u> knowing values of T on a basis determines T.

<u>Proof</u> $v \in V$, then $v = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$ <u>uniquely</u>.

so $T(\overset{v}{\cancel{x}}) = x_1 T(v_1) + \dots x_n T(v_n)$ $\square$.

<u>Example</u> P_n = degree n polynomials.

P_n standard basis $\{t^n, t^{n-1}, \dots, t, 1\}$.

<u>integration</u> (with additive constant 0)

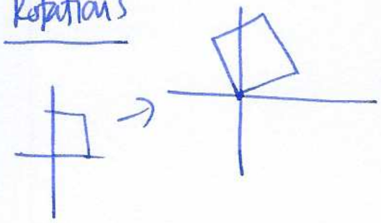
$P^n \longrightarrow P^{n+1}$ $\qquad a_n t^n + a_{n-1} t^{n-1} + \cdots a_1 t + a_0 \mapsto \frac{a_n t^{n+1}}{n+1} + \frac{a_{n-1} t^n}{n} + \cdots + \frac{a_1 t^2}{2} + a_0 t$

$p(t) \mapsto \int p(t)\,dt + 0$

special case $I: P^2 \to P^3$: $\qquad a_2 t^2 + a_1 t + a_0 \longmapsto \frac{a_2 t^3}{3} + \frac{a_1 t^2}{2} + a_0 t + 0$

$$I = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

<u>Rotations</u>



$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

<u>note</u> $R_\phi \circ R_\theta$ should be $R_{\phi+\theta}$ check:

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{bmatrix}$$

trig addition formulae : $= \begin{bmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{bmatrix}$

<u>General fact</u> $\quad U \xrightarrow{S} V \xrightarrow{T} W$ linear maps

$u \mapsto S(u) \mapsto T(S(u))$

pick bases $\{u_1, \dots u_\ell\}$ $\{v_1, \dots, v_m\}$ $\{w_1, \dots w_n\}$

then $\quad S \leftrightarrow \underset{m \times \ell}{A} \qquad T \leftrightarrow \underset{n \times m}{B}$.

$x \longmapsto S(x) \longmapsto T(S(x))$

$\ell \times 1 \qquad \underbrace{\underset{m\times\ell \;\; \ell\times 1}{Ax}}_{m \times 1} \longmapsto \underbrace{\underset{n\times m \;\; m\times\ell \;\; \ell\times 1}{BAx}}_{n \times 1}$.

<u>Fact</u> composition of linear functions corresponds to matrix multiplication.