Proof $V \leq \mathbb{R}^n$, check $V^\perp$ is a subspace.

addition: let $x, y \in V^\perp$ then $x.v = 0$ for all $v \in V$
$y.v = 0$ for all $v \in V$

$(x+y).v = x.v + y.v = 0 + 0 = 0$ for all $v \in V$.

scalar multiplication: $cx.v = c(x.v) = c.0 = 0$ for all $v \in V$. $\square$.

Thm (4 subspaces are orthogonal complements). A $m \times n$.

$$N(A) = \text{row}(A)^\perp \leq \mathbb{R}^n \quad \text{also} \quad N(A)^\perp = \text{row}(A)$$
$$N(A^T) = \text{col}(A)^\perp \leq \mathbb{R}^m \quad\quad N(A^T)^\perp = \text{col}(A).$$

Proof recall: $\left.\begin{array}{l}\dim(N(A)) = n-r \\ \dim(\text{row}(A)) = r\end{array}\right\}$ orthogonal vectors independent! so $\dim(\text{span}\{N(A), \text{row}(A)\}) \geq n$

similarly $\left.\begin{array}{l}\dim(N(A^T)) = m-r \\ \dim(\text{col}(A)) = r\end{array}\right\}$ so $\text{span}\{N(A^T), \text{col}(A)\} = \mathbb{R}^m$.

Check suppose $x \in \text{row}(A)^\perp$, choose orthogonal basis for $N(A)$ $a_1, \dots a_{n-r}$
$\text{row}(A)$ $b_{n-r}, \dots, b_n$

then $a_1, \dots, a_{n-r}, b_1, \dots, b_r$ is a basis for $\mathbb{R}^n$

so $x = x_1 a_1 + \dots + x_{n-r} a_{n-r} + x_{n-r+1} b_{n-r+1} + \dots x_n b_n. = a + b.$

~~now consider Ax~~ note all $\cancel{b_i}\ x_i = 0$ for $i \geq n-r+1$ as $x.b_i = 0 = x_1 \underbrace{a_1.b_i}_{=0} + \dots x_{n-r} \underbrace{a_{n-r}.b_i}_{=0}$

$+ x_{n-r+1} \underbrace{b_{n-r+1}.b_i}_{=0} + \dots x_i \underbrace{b_i.b_i}_{\|b_i\|^2} + \dots + x_n \underbrace{b_n.b_i}_{=0} = 0 \Rightarrow x_i = 0$

so $x = a + 0$, i.e. $x \in N(A)$, as required. $\square$.

More generally Thm if V, W orthogonal subspaces of $\mathbb{R}^n$, and $\dim V + \dim W = n$, then $V = W^\perp$ and vice versa.

Proof let $v_1, \dots v_a$ be orthogonal basis for V $a+b=n$
$w_1, \dots w_b$ W

then $v_1, \dots, v_a, w_1, \dots, w_b$ is orthogonal basis for $\mathbb{R}^n$.

so any $x \in V^\perp$ is $x = x_1 v_1 + \dots x_a v_a + y_1 w_1 + \dots + y_b w_b = \overset{\in V}{a} + \overset{\in W}{b}$

but $x \cdot v_i = 0$ so $x_1 \underbrace{v_1 \cdot v_i}_{=0} + \dots x_i \underbrace{v_i \cdot v_i}_{=\|v_i\|} + \dots + x_a \underbrace{v_a \cdot v_i}_{=0} + y_1 \underbrace{w_1 \cdot v_i}_{=0} + \dots + y_b \overbrace{w_b \cdot v_i}^{=0} = 0$

$\Rightarrow x_i = 0$ for $1 \le i \le a$.

so $x = 0 + b \in W$. so $W = V^\perp$. $\square$.

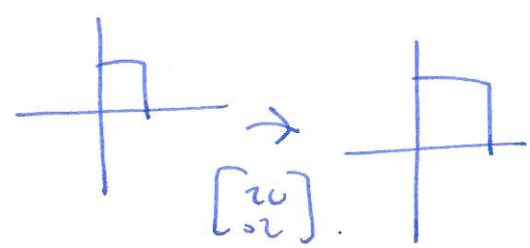
Linear transformations A $m \times n$ matrix $A: \mathbb{R}^n \to \mathbb{R}^m$
$\underbrace{x}_{n\times 1} \mapsto \underbrace{Ax}_{m \times 1}$

Examples • $A: \overset{\mathbb{R}^1}{\mathbb{R}} \to \mathbb{R}$
$x \mapsto ax$

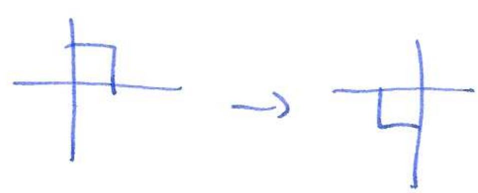
$\mathbb{R}^2$: $A = I$ $\underline{x} \mapsto \underline{x}$

$A = 0$ $x \mapsto 0$

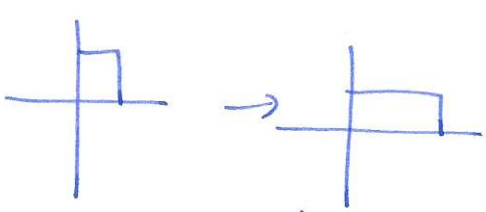
$A = cI = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ expansion $c > 1$
contraction $0 < c < 1$



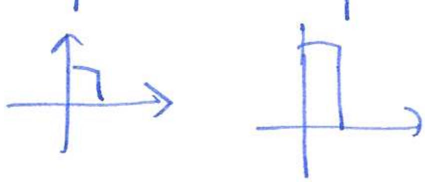
$c = -1$



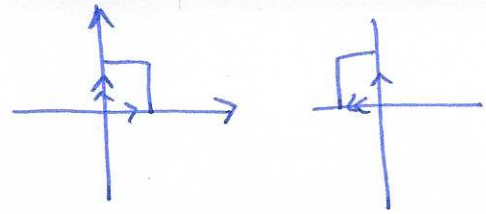
$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.



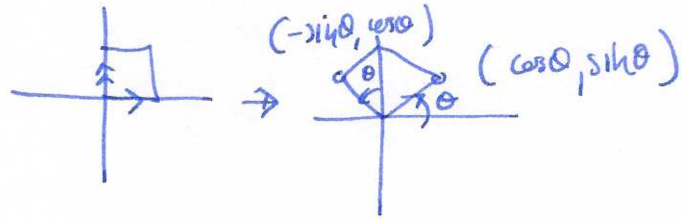
$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$



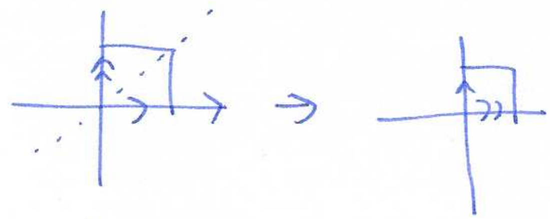
- $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotation by $\pi/2$ $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$



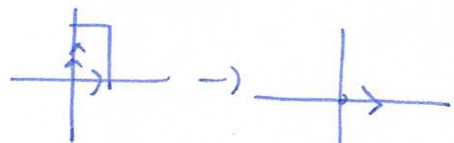
- general rotation by θ : $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



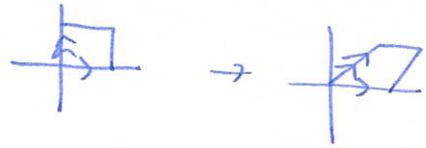
- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflection in $x=y$ $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$



- $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ projection $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$



- $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ shear $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+y \\ y \end{bmatrix}$



<u>Observations</u> $\underset{m \times n}{A} : \mathbb{R}^n \to \mathbb{R}^m$

- $A0 = 0$ always sends zero vector to zero vector
- $A(cx) = c(Ax)$
- $A(x+y) = Ax + Ay$

<u>Defn</u> V, W vector spaces, $T : V \to W$ is a <u>linear map</u> / function transformation if
- $T(cx) = c(Tx)$
- $T(x+y) = T(x) + T(y)$

<u>Examples</u> $\underset{m \times n}{A} : \mathbb{R}^n \to \mathbb{R}^m$
$x \mapsto Ax$

<u>Fact</u> : every linear map can be described as a matrix