Existence of inverses

recall : if A has a left inverse and a right inverse, they are equal

fact : A has inverse iff A is square $n\times n$ and $\mathrm{rank}(A)=n$

one-sided inverses

Example $\underset{2\times 3}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \\ \cdot & \cdot \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ← not unique!

Fact $\underset{m\times n}{A}$ $\mathrm{rank}(A)=r$

if $r=m$ then there is a right inverse $A^T(AA^T)^{-1}$ check!

$r=n$ then there is a left inverse $(A^TA)^{-1}A^T$ check!

Q: when is AA^T invertible? $\underset{m\times n\ n\times m}{A\,A^T}$ $m\times m$ $\mathrm{rank}(A)=m$

$\underset{n\times m\ m\times n}{A^T A}$ $n\times n$ $\mathrm{rank}(A)=n$

two sided inverses

$\underset{m\times n}{A}$ has a two sided inverse iff $m=n=\mathrm{rank}(A)$

equivalent conditions

- columns of A span $\mathbb{R}^n$, so $Ax=b$ always has a solution
- the rows of A span $\mathbb{R}^n$
- the rows are linearly independent
- elimination gives $PA=LDU$ with n pivots
- $\det(A)\neq 0$ how to find $[A|I] \xrightarrow{\text{rowops}} [I|A^{-1}]$

Linear transformations

A $m\times n$ matrix $A:\mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\underset{n\times 1}{x} \longmapsto \underset{m\times 1}{Ax}$

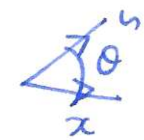
Orthogonal vectors and subspaces

$\mathbb{R}^n$: length of vector $\langle x_1, \dots, x_n \rangle$ $\quad \|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \|x\|^2 = x_1^2 + x_2^2 + \dots + x_n^2 = x^T x$$

recall: dot product $x \cdot y = x^T y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

$$= \|x\| \|y\| \cos\theta$$



two vectors are orthogonal if $x \cdot y = 0$

A set of vectors $\{v_1, \dots, v_k\}$ is orthogonal if $v_i \cdot v_j = 0$ for all $i \neq j$.

Lemma orthogonal vectors are linearly independent

Proof suppose $c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0$

then $(c_1 v_1 + c_2 v_2 + \dots + c_k v_k) \cdot v_i = 0 \cdot v_i = 0$

$$c_1 \underbrace{v_1 \cdot v_i}_{=0} + c_2 \underbrace{v_2 \cdot v_i}_{=0} + \dots + c_i \underbrace{v_i \cdot v_i}_{= \|v_i\|} + \dots + c_k \underbrace{v_k \cdot v_i}_{=0} = 0$$

$c_i \|v_i\| = 0 \Rightarrow c_i = 0$ for all i. $\square$.

Special bases $\{v_1, \dots, v_d\}$ orthogonal basis if $v_i \cdot v_j = 0$ for all $i \neq j$

$\{v_1, \dots, v_d\}$ orthonormal basis if $v_i \cdot v_j = 0$ for all $i \neq j$ and $\|v_i\| = 1$ for all i.

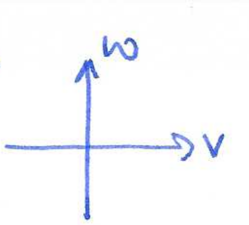
Examples $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\}$.

Orthogonal subspaces

subspaces V, W are orthogonal if $v \cdot w = 0$ for all $v \in V$ and $w \in W$

Examples $\{0\}$ orthogonal to every subspace.

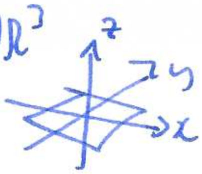
$\mathbb{R}^2$



V basis $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

W basis $\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$

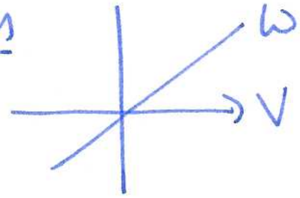
$\mathbb{R}^3$



V basis $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$.

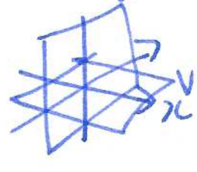
W basis $\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

Nonexamples



V basis $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

W basis $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$



V basis $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

W basis $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

Thm (~~Fundament thm of~~ Orthogonality for 4 subspaces). A $m\times n$

The row space is orthogonal to the null space/kernel ($\subseteq \mathbb{R}^n$)

The column space is orthogonal to the left null space/cokernel ($\subseteq \mathbb{R}^m$)

Proof ① suppose $x \in N(A)$, then $Ax = 0$ $\quad A\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = \begin{bmatrix}0\\\vdots\\0\end{bmatrix}$

i.e. x is orthogonal to all of the rows of A, and so is orthogonal to all linear combinations of the rows. $(c_1 r_1 + c_2 r_2 + \cdots + c_m r_m)\cdot x$

$$= \underbrace{c_1 r_1 x}_{=0} + \underbrace{c_2 r_2 x}_{=0} + \cdots + \underbrace{c_m r_m x}_{=0} = 0$$ so $N(A) \perp Row(A)$

now suppose $y \in N(A^T)$ so $y^T A = 0$ $\quad [y_1 \cdots y_n] A = \begin{bmatrix}0\\\vdots\\0\end{bmatrix}$

i.e. x is orthogonal to the columns of A, and hence to all linear combinations of the columns, so $N(A^T) \perp col(A)$.

② suppose $x \in N(A)$ so $Ax = 0$

let $v \in Row(A)$, so $v = A^T y$ for some y

then $v^T x = (A^T y)^T x = y^T A x = y^T 0 = 0$. $\square$.

Def^n let $V \subseteq \mathbb{R}^n$ be a subspace, then the collection of all vectors perpendicular to V is a subspace, called the orthogonal complement $V^\perp$ ("V perp").