this shows $A \rightsquigarrow U \leftarrow$ (reduced) echelon form then $\mathrm{Row}(A) = \mathrm{Row}(U)$

observation: the pivot rows of U form a basis for $\mathrm{Row}(U) = \mathrm{Row}(A)$

Example
$$\underset{A}{\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}} \rightsquigarrow \underset{U}{\begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$
basis: $\{ \langle 1,3,3,2 \rangle, \langle 0,0,3,3 \rangle \}$.

check: · they span $\mathrm{Row}(U)$ as zero rows contribute nothing

· they are linearly independent. suppose $c_1 r_1 + c_2 r_2 + \cdots + c_k r_k = 0$

r_1 contains a pivot with zeros below $\Rightarrow c_1 = 0$, then r_2 contains a pivot with zeros below $\Rightarrow c_2 = 0$, etc. so
$$\begin{aligned} \dim(\mathrm{Row}(A)) &= \dim(\mathrm{Row}(U)) \\ &= \mathrm{rank}(A) \\ &= \#\text{pivots} = r. \end{aligned}$$

Nullspace | Kernel

observation: row operations do not change the solution set to the equations, because they are reversible.

$Ax = 0$
$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} x = \begin{bmatrix} r_1 x \\ r_2 x \\ r_3 x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad B = \begin{bmatrix} r_1 \\ r_2 - r_1 \\ r_3 + 4r_1 \end{bmatrix} x = \begin{bmatrix} r_1 x \\ (r_2 - r_1)x \\ (r_3 + 4r_1)x \end{bmatrix} = \underline{0}$$

or $L_1 A = B$ with L_1 invertible $A = L_1^{-1} B$

so $Bx = L_1 A x = 0$ $\qquad Bx = 0 \Rightarrow L_1^{-1} B x = 0 \Rightarrow Ax = 0$.

so $N(A) \subseteq \mathrm{Null}(B)$ $\qquad \mathrm{Null}(B) \subseteq \mathrm{Null}(A)$.

$\Rightarrow \mathrm{null}(A) = \mathrm{null}(B)$.

observation: to find a basis for $N(A)$ do back substitution in U.

get a basis vector for each free variable, so $\dim(N(A)) = \#\text{cols} - \#\text{pivots}$
$= n - r$

recall in example $s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ they span as they give all solutions
they are lin. indep as U is upper triangular.

Column space $C(A)$

$\underset{m\times n}{A}\ \underset{n\times 1}{x}$ $\qquad A : \mathbb{R}^n \to \mathbb{R}^m$

note: $C(A)$ is also the image of A

$= \{ Ax \in \mathbb{R}^m \mid x \in \mathbb{R}^n \}$ as $\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 c_1 + x_2 c_2 + \cdots + c_n x_n$

warning row operations change the column space

Example $\overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \rightsquigarrow \overset{U}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}$

claim the columns of A corresponding to pivots of U are a basis for $C(A)$.

so $\dim(C(A)) = \#\text{pivots} = \operatorname{rank}(A) = r = \dim(\operatorname{Row}(A))$

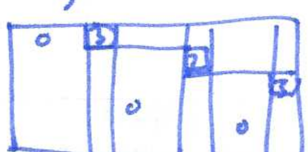
Proof observation: $Ax = 0 \Leftrightarrow Ux = 0$ $\quad (N(A) = N(U))$

a linear dependence between the columns of A is $x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = 0$

i.e. $\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ i.e. $Ax = 0$, but then $Ux = 0$, so this gives a linear dependence between the columns of U, and vice versa

The pivot columns of U are a basis for $C(U)$

so the corresponding cols of A are a basis for $C(A)$



$\square$

Left null space $N(A^T)$

i.e. $y^T A = 0$ $\qquad \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix} = y_1 r_1 + y_2 r_2 + \cdots + y_m r_m = 0$

i.e. linear combination of rows giving 0

recall: $PA = LU$

$\underbrace{L^{-1}P}\, A = U =$ 

bottom $m-r$ rows of $L^{-1}P$ are a basis for $N(A^T)$

alternate method: just solve $A^T y = 0$

Fundamental theorem of linear algebra part I

$\dim(C(A)) = r$ $\qquad \dim(\operatorname{Row}(A)) = r$

$\dim(N(A)) = n - r$ $\qquad \dim(N(A^T)) = m - r$