**Observation** $A_{m\times n}$ matrix with $n>m$

row reduce $$m\left[\underbrace{\qquad\qquad}_{m}\vdots\underbrace{\qquad\qquad}_{n-m}\right]$$ #pivots $\leq m < n$ so free variables so columns are **dependent**

**Corollary** a set of $n$ vectors in $\mathbb{R}^m$ with $n>m$ **must** be dependent.

**Example** $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

**spanning a vector space**

$V$ vector space $v_1, \dots, v_k$ vectors in $V$

**Defn** we say $v_1, \dots, v_k$ **span** $V$ if every vector in $V$ is a linear combination $c_1v_1 + c_2v_2 + \dots + c_kv_k$ of $v_1, \dots, v_k$.

**Remark** this linear combination need not be unique.

**Example** ① $v_1 = \langle 1,0,0\rangle$
$v_2 = \langle 0,1,0\rangle$
$v_3 = \langle -2,0,0\rangle$
span the $xy$-plane in $\mathbb{R}^3$.

② $\operatorname{col}(A)$ is spanned by the columns of $A$

③ $V$ is spanned by all vectors in $V$

④ the standard unit vectors span $\mathbb{R}^n$
$e_1 = \langle 1,0,\dots,0\rangle$
$e_2 = \langle 0,1,0,\dots 0\rangle$
$\vdots$
$e_n = \langle 0,0,\dots,0,1\rangle$

**Example** if $v_1, \dots, v_k$ span $V$, then adding any collection of vectors to $v_1, \dots, v_k$ also spans $V$

special spanning sets: minimal ones, can't throw any vectors away

Defⁿ A basis for V is a collection of vectors which are

① linearly independent

② span V

Observations · if $b_1, \dots, b_k$ is a basis for V, then every vector $v \in V$ is a linear combination of basis vectors, i.e. $v = c_1 b_1 + c_2 b_2 + \cdots c_k v_k$.

· this linear combination is unique.

suppose $v = a_1 b_1 + a_2 b_2 + \cdots + a_k b_k = c_1 b_1 + c_2 b_2 + \cdots + c_k b_k$

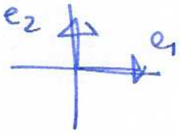
then $v - v = a_1 b_1 + \cdots + a_k b_k - c_1 b_1 - \cdots - c_k b_k$

$0 = (a_1 - \cancel{b}c_1) b_1 + (a_2 - b_2) c_2 + \cdots + (a_k - c_k) b_k$

the $b_i$ are linearly independent so the only way to get zero vector is if $a_i - c_i = 0$ for all i, ~~a~~ i.e. $a_i = c_i$ for all i.

Examples of bases

· $\mathbb{R}^2$ $\{e_1, e_2\} = \{\langle 1,0 \rangle, \langle 0,1 \rangle\}$



· any pair of vectors which span $\mathbb{R}^2$, i.e. are not parallel.

e.g. $\{\langle 1,0 \rangle, \langle 1,1 \rangle\}$, $\{\langle 2,3 \rangle, \langle 4,-1 \rangle\}$.

· there are infinitely many different bases for $\mathbb{R}^2$ (a basis is not unique)

· $\mathbb{R}^n$: $\{e_1, e_2, \dots, e_n\}$

Nonexamples: any set containing 0 zero vector.

Defⁿ Any two bases for V have the same number of vectors. The number of vectors in a basis is called the dimension of V.

Examples $\dim(\mathbb{R}^2) = 2$ $\dim(\mathbb{R}^n) = n$.

Thᵐ Let V be a vector space with bases ~~B~~, $v_1, \dots, v_m$, and $w_1, \dots w_n$ then $m = n$.

Proof wlog assume $n > m$. The $v_i$ are a basis, so they span V, so every $w_j$ is a linear combination of the $v_i$, i.e.

$$w_1 = a_{11} v_1 + a_{21} v_2 + \cdots + a_{m1} v_m$$
$$w_2 = a_{12} v_1 + a_{22} v_2 + \cdots + a_{m2} v_m$$
$$\vdots$$
$$w_n = a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{mn} v_m$$

let $W = [w_1 \; w_2 \; \dots \; w_n]$ $\quad V = [v_1 \; \dots \; v_n]$ $\quad A = \underset{n}{\begin{bmatrix} a_{11} \dots & a_{1n} \\ \vdots & \\ a_{m1} \dots & a_{mn} \end{bmatrix}} m$

then $W = VA$ $\quad$ A is $m \times n$, so A has at most m pivots $< n$

$\Rightarrow$ there is a free variable after row reduction $\Rightarrow$ non-trivial solution to

$Ax = 0$ $\quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad$ so $VAx = V\underline{0} = 0$

so $W\underline{x} = 0$

$\Rightarrow$ $x_1 w_1 + x_2 w_2 + \cdots + x_n w_n = 0$, not all $x_i = 0$ # $w_i$ linearly independent □

therefore $m = n$.

**Thm.** · Any linearly independent set can be extended to a basis, by adding more vectors if necessary

· Any spanning set can be reduced to a basis, by discarding vectors if necessary.

**Remarks** A basis is a maximal independent set

it is also a minimal spanning set

· If V has dimension n : at most n vectors may be linearly independent

at least n vectors are needed to span the space

**Examples** · quadratic polynomials $P_2$ $\quad ax^2 + bx + c$ $\quad$ basis $\{1, x, x^2\}$

dim = 3.

· 2x2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ dim = 4.

## The four fundamental subspaces

Given a matrix A there are 4 natural subspaces built from $\underset{m\times n}{A}$

- column space col(A) spanned by columns of A $\dim(\text{col}(A)) = \text{rank}(A) = r$
- null space $\text{null}(A) = \{x \mid Ax = 0\}$ dimension $n-r$ (kernel)
- row space spanned by rows of A, $\text{col}(A^T)$, dimension $= r = \text{rank}(A)$
- left nullspace / cokernel, all vectors y s.t. $y^T A = 0$, dimension $m-r$

$= \text{null}(A^T)$ as $(y^T A)^T = A^T (y^T)^T = A^T y = 0$.

$\underset{m\times n}{A}$ nullspace null(A), row space, subspaces of $\mathbb{R}^n$

left nullspace $\text{null}(A^T)$, col space, subspaces of $\mathbb{R}^m$

Example $A = \begin{bmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\underset{m\times n}{2\times 3}$ $r = 1$

col(A) spanned by $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ dim $= 1 = r$

null(A) $s\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\0\\1\end{bmatrix}$ spanned by $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

dim $\underset{n-r}{3-1} = 2$.

row(A) spanned by $\langle 0,1,0\rangle$ dim $= r = 1$

cokernel, $\text{null}(A^T)$ spanned by $\langle 0,1\rangle$, dim $m - r = 2 - 1 = 1$.

### General case / how to find bases for these subspaces

**Row space**

observation: row operations do not change the row space, because they are reversible.

$$\begin{matrix}\textcircled{1}\\ \textcircled{2}\\ \textcircled{3}\end{matrix}\begin{bmatrix} A \end{bmatrix} \rightsquigarrow \begin{matrix}\textcircled{1}\\ \textcircled{2}+3\textcircled{1}\\ \textcircled{3}-4\textcircled{1}\end{matrix}\begin{bmatrix} B \end{bmatrix}$$

↗ row space, linear combination of rows 1,2,3.

↑ row space, linear combinations of ①, ②+3①, ③−4①

so contained in row A, i.e. $\text{row}(B) \le \text{row}(A)$

but ①,②,③ are linear combinations of rows of B,

so $\text{row}(A) \le \text{row}(B)$

$\Rightarrow$ $\text{row}(A) = \text{row}(B)$.