- can choose x_p to have all free variables zero
- x_n are the solutions to $Ax=0$ } there is an $(n-r)$ dimensional space of these.

<u>Example</u> $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & \\ 2 & 4 & 8 & 12 & \\ 3 & 6 & 7 & 13 & \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & \\ 0 & 0 & 2 & 2 & \\ 0 & 0 & -2 & -2 & \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & \\ 0 & 0 & 2 & 2 & \\ 0 & 0 & 0 & 0 & \end{array}\right]$

<u>Linear independence, basis, dimension</u>

$\underset{m\times n}{A}$ $\mathrm{col}(A)$ column space, $\mathrm{null}(A)$ null space } vector spaces ← a vector space has a "size" called the <u>dimension</u>

$\#\text{pivots} = r = \mathrm{rank}(A) = \dim \mathrm{col}(A)$

$n-r = \mathrm{nullity}(A) = \dim \mathrm{null}(A)$

<u>Recall</u> : a linear combination of the vectors $v_1, v_2, \dots, v_k$ is a vector of the form $c_1v_1 + c_2v_2 + \cdots + c_kv_k$

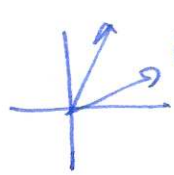
<u>Defn</u> A set of vectors $v_1, \dots, v_k$ is <u>linearly independent</u> if

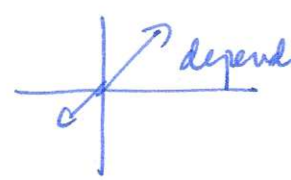
$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$ implies $c_1=0, c_2=0, \dots, c_k=0$ i.e. "the only way to get zero vector is to have every coefficient be zero".

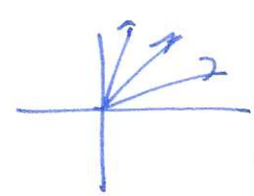
If there is any ~~\$~~ non-trivial way to get zero, we say the vectors are <u>linearly dependent</u>, i.e. $\exists\, c_1, \dots, c_k$ not all zero s.t. $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$

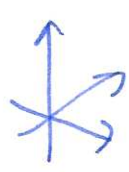
<u>Observation</u> A set containing the zero vector is <u>always</u> linearly dependent as if $v_i = 0$ then $c_i = 1$, all other $c_j = 0$ give ~~$c_i v$~~ $1.0 = 0$ vector.

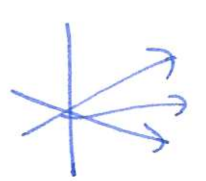
<u>Examples</u>


independent


dependent


dependent.


independent


dependent

Example $\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$ columns are dependent $C2 = 3C1$
rows are dependent $R3 = 2R2 - 5R1$.

Example $\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ columns are independent
rows are independent

Q: how do we check this? $C(A)$ column space $A = [c_1\ c_2\ \cdots\ c_n]$
are the columns independent? want $a_1c_1 + a_2c_2 + \cdots a_nc_n = 0$
i.e. solve $Ax = 0$: if only solution is $x = 0$ columns are <u>independent</u>
(i.e. $\text{null}(A) = \{0\}$).

if any non-zero solution columns are <u>dependent</u>
(i.e. $\text{null}(A)$ bigger than $\{0\}$)

row reduce:
(A) $\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

all columns can be made from sums of pivot columns
- pivot columns are independent
- extra columns give dependent vectors.

(B) $\begin{bmatrix} \boxed{3} & 4 & 2 \\ 0 & \boxed{1} & 5 \\ 0 & 0 & \boxed{2} \end{bmatrix}$ 3 pivots $\text{null}(A) = \{0\}$, columns independent.

<u>note</u> row reduction shows: (A) has <u>dependent</u> rows (get $[0\ 0\ 0\ 0]$)
(B) has independent rows.

<u>Summary</u> The $r = \text{rank}(A)$ non-zero rows of U and R are independent
The $r = \text{rank}(A)$ columns with pivots of U and R are independent

<u>Example</u> $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n \times n}$ $\leftarrow \text{rank}(I) = n$
all rows/cols independent.