Factorization even if A is not square, can still record the row operations in a matrix, so if we do $A \rightsquigarrow U$ row echelon form

then $\underset{m\times n}{A} = \underset{\substack{m\times m\\ \text{lower}\\ \text{triangular}}}{L} \; \underset{\substack{m\times n\\ \text{upper triangular}}}{U}$ as before.

Solving $Ax=b$

- case $b=0$ is special, as row operations don't change zero column.

$$Ax=0 \quad \begin{bmatrix} 1 & 3 & 3 & 2 & 0 \\ 2 & 6 & 9 & 7 & 0 \\ -1 & -3 & 3 & 4 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

naive method (works for solving $Ax=b$, only need to solve for one value of b)

just do row operations on: $\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 2 & 6 & 9 & 7 & b_2 \\ -1 & 3 & 3 & 4 & b_3 \end{array}\right]$

better method (when you need to solve for many different b's).

$Ax=b$, find $A=LU$, so $LUx=b$.

can solve: $Lc=b$, then $Ux=c$ (or solve $Ux=L^{-1}b$).

Example
$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \underset{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}{\rightsquigarrow} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix} \underset{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}{\rightsquigarrow} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix}\right).$$

solve $Ax = b$ $\quad L\underbrace{Ux}_{c} = b.$ $\quad Lc = b$ $\quad Ux = c$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$c_1 = b_1$

$2c_1 + c_2 = b_2$ $\qquad c_2 = b_2 - 2b_1$

$-c_1 + 2c_2 + c_3 = b_3$ $\qquad c_3 = b_3 + b_1 - 2b_2 + 4b_1$
$= 5b_1 - 2b_2 + b_3.$

$Ux = c$ $\quad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ has sol^n iff consistent

i.e. need $c_3 = 5b_1 - 2b_2 + b_3 = 0$!

(try $\underline{b} = \langle 1, 5, 5 \rangle$)

Observation $Ax = b$ can be solved iff b lies in the column space of A $\mathrm{col}(A)$. Warning not $\mathrm{col}(U)$!!!

even though A has 4 cols, the column space is 2 dimensional, as the columns without pivots are the sums of the other columns

$$\begin{array}{cccc} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ c_1 \quad c_2 \quad c_3 \quad c_4 \end{array}$$

$c_2 = 3c_1$

$c_4 = c_3 - c_1$

so we have two descriptions of the column space $\mathrm{col}(A)$:

- all linear combinations of c_1, c_2, c_3, c_4 (in fact c_1, c_3)
- all solutions to $5b_1 - 2b_2 + b_3 = 0$.

Example $b \in \mathrm{col}(A)$ solve $Ax = b$ $\quad b = \langle 1, 5, 5 \rangle$

$\left.\begin{array}{l} Ax = b. \\ LUx = b \end{array}\right\}$ $\quad \begin{array}{l} Lc = b \rightarrow \\ Ux = c \end{array}$ $\quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$

$c_1 = 1$

$2c_1 + c_2 = 5$

$c_2 = 3.$

$$\begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

free var $\qquad$ free var

$-c_1 + 2c_2 + c_3 = 5$

$-1 + 6 + c_3 = 5$

$c_3 = 0$

back substitution: $x_4 = s$ $\qquad x_4 = s$

$3x_3 + 3x_4 = 3$ $\qquad x_3 = 1 - s$

$x_2 = t$ $\qquad x_2 = t$

$x_1 + 3x_2 + 3x_3 + 2x_4 = 1$ $\qquad x_1 = 1 - 3t - 3(1-s) - 2s = -2 - 3t + s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2-3t+s \\ t \\ 1-s \\ s \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{\cancel{t}\,s \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{x_n}$$

<u>key observation</u> ①

the solutions are all of the form $x_p + x_n$

some particular solution → x_p ; x_n ← solutions to $Ax = 0$

Note: if x_p and x_p' are two different solutions ~~to~~ then $A(x_p - x_p') = Ax_p - Ax_p' = b - b = 0$

so doesn't matter which x_p you choose.

<u>Reduced equations</u> $\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

<u>key observation</u> ② $\underset{m \times n}{A} x = b$

if there are <u>r pivots</u>, then there are <u>n-r free vars.</u>

<u>Def$^{\underline{n}}$</u> the number of pivots is the <u>rank</u> of A

Furthermore, the last m-r rows of U (and R) are zero!

<u>summary</u> $Ax = b$ row reduces to $Ux = c$ and $Rx = d$ with r pivots (rank(A) = r) Then:

- the last m-r rows of U and R are zero, so there is a solution iff the last m-r entries of c and d are zero
- the complete solution set is $x = x_p + x_n$