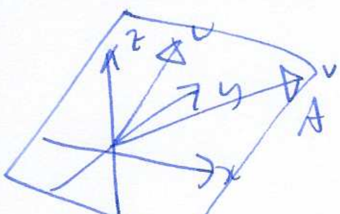
Column spaces

Let A be a matrix. The column space of A consists of all linear combinations of the columns of A, written $C(A)$

Example $A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}$ linear combination of A: $c_1 \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$

e.g. $\begin{bmatrix} 2 \\ 10 \\ 4 \end{bmatrix} + \begin{bmatrix} -0 \\ -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} \in C(A)$

observation the linear system $Ax = b$ has a solution iff the vector b is a linear combination of the cols of A, i.e. $b \in C(A)$.



Notation A $m \times n$ matrix then $C(A)$ is the column space consisting of all linear combinations of the columns.

claim $C(A)$ is a subspace $\quad A = [a_1 \ a_2 \ \cdots \ a_n]$

Proof suppose $b_1 = c_1 a_1 + c_2 a_2 + \cdots + c_n a_n$

$b_2 = d_1 a_1 \ d_2 a_2 + \cdots + d_n a_n$

then $b_1 + b_2 = (c_1 + d_1) a_1 + (c_2 + d_2) a_2 + \cdots + (c_n + d_n) a_n$

alternatively: suppose $Ac = b_1$, $Ad = b_2$ then $A(c+d) = b_1 + b_2$

if $b = c_1 a_1 + c_2 a_2 + \cdots + c_n a_n$ then $cb = c c_1 a_1 + c c_2 a_2 + \cdots + c c_n a_n$

or $cAx = cb$

$\quad \,\, \|$

$A(cx) = cb$ □.

Example A 5×5 matrix how big can $\mathrm{Col}(A)$ be?

$A=0$ then $\mathrm{Col}(A)=\{0\}\subset\mathbb{R}^5$

$A=I$ then $\mathrm{col}(A)=\mathbb{R}^5$

Observation if A is $n\times n$ and non-singular, then $\mathrm{col}(A)=\mathbb{R}^n$

because $Ax=b$ has a (unique) solution for every $b\in\mathbb{R}^n$

Null spaces

Defn The nullspace of a matrix A consists of all vectors x such that $Ax=0$. if A is $m\times n$, then $\mathrm{null}(A)\subseteq\mathbb{R}^m$

Claim the nullspace of A is a vector space.

Proof addition: if $Ax=0$ and $Ay=0$ then $A(x+y)=Ax+Ay$
$=0+0=0$

scalar multiplication: if $Ax=0$ then $A(cx)=c(Ax)=c0=0$ $\square$

warning if $b\neq0$ then the solution set to $Ax=b$ is not a vector space!
(does not contain 0!).

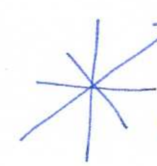
examples (1) $\begin{bmatrix}1&0\\5&4\\2&4\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ row reduce: $\begin{bmatrix}1&0\\0&4\\0&0\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

$u=0$
$4v=0$ $\Rightarrow\langle u,v\rangle=\langle 0,0\rangle$

so $\mathrm{null}(A)=\{0\}$.

(2) $\begin{bmatrix}1&0&1\\5&4&9\\2&4&6\end{bmatrix}\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ so $\mathrm{null}(A)$ contains $\langle 1,1,-1\rangle$
hence any multiple of $\langle 1,1,-1\rangle$.

Q: anything else?



line of solution.

$\sim\begin{bmatrix}1&0&1\\0&4&4\\0&4&4\end{bmatrix}\sim\begin{bmatrix}\boxed{1}&0&1\\0&\boxed{1}&1\\0&0&0\end{bmatrix}$

$x+z=0$ $x=-t$

$y+z=0$ $y=-t$

$z=t$

$z=t$ so solution is $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-t\\-t\\t\end{bmatrix}=t\begin{bmatrix}-1\\-1\\1\end{bmatrix}$.

Solving $Ax=0$ and $Ax=b$

so far: special case $Ax=b$ A $n\times n$, full set of n pivots, unique solution $x=A^{-1}b$

observations ① consider $Ax=b$ and suppose • x_p is a solution $Ax_p=b$

• x_n is in the nullspace null(A)

i.e. $Ax_n=0$

then x_p+x_n is a solution

check: $A(x_p+x_n) = Ax_p + Ax_n = b+0=b$.

② $0x=b$ has no solutions, unless $b=0$. $b\neq 0$ inconsistent.

Examples

$$x+y=2$$
$$2x+2y=4$$

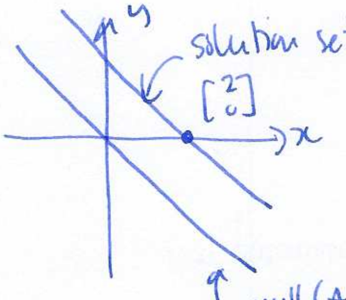
$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$y=t$

$x+y=2 \to x+t=2$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2-t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



solution set $=$ null$(A)+x_p$

$t\begin{bmatrix} -1 \\ 1 \end{bmatrix}+\begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

null(A), line through 0 in direction $\langle -1,1\rangle$.

$$x+y=2$$
$$2x+2y=5$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right]$$

$0x+0y=1$ #

no solutions.

<u>Example</u> $\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

we can always row reduce to put the matrix in <u>row-echelon form</u> 5

① the pivots are the first non-zero entries in their rows.
② below each pivot is a column of zeros.
③ each pivot lies to the right of the pivot above

we can do two extra steps to get <u>reduced row-echelon form</u> $\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

④ make all pivots 1 by dividing each row by its pivot
⑤ row reduce upwards to get zeros <u>above</u> the pivots $\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

<u>Note</u> solutions to $Ax=0$ are the same as solutions to $Rx=0$ as each row operation is <u>reversible</u>, now solve by back substitution.

$$\begin{bmatrix} \boxed{1} & 3 & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

each column without a pivot corresponds to a <u>free variable</u>

$\uparrow$ free var s $\uparrow$ free var t

$w = t$

$z + w = 0 \Rightarrow z = -t$

$y = s$

$x + 3y - w = 0 \quad x = -3s + t$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -3s+t \\ s \\ -t \\ t \end{bmatrix} = \underbrace{t \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{this describes the nullspace (2d)}}$$

<u>Observation</u> If $Ax=0$ has more unknowns than equations $(m > n)$ then it has at least one free variable, i.e. there is at least one <u>non-trivial</u> $(\neq 0)$ solution.

<u>Preview</u> # pivots = dimension of $col(A)$
free variables = dimension of $null(A)$